ON DISCRETE SUBORDINATION OF POWER BOUNDED AND RITT OPERATORS

ALEXANDER GOMILKO AND YURI TOMILOV

Abstract. By means of a new technique, we develop further a discrete subordination approach to the functional calculus of power bounded and Ritt operators initiated by N. Dungey in [23]. This allows us to show, in particular, that (infinite) convex combinations of powers of Ritt operators are Ritt. Moreover, we provide a unified framework for several main results on discrete subordination from [23] and answer a question left open in [23]. The paper can be considered as a complement to [31] for the discrete setting.

1. Introduction

The aim of this paper is to initiate a study of permanence and "improving" properties of discrete subordination for bounded operators parallel in a sense to an investigation of subordination for C_0-semigroups realized in our recent paper [31]. A discrete subordination in the abstract setting has not received a proper attention in the literature, and we are not aware of any relevant works apart from [23] and [7]. At the same time, our paper can be regarded as a contribution to understanding of permanence and improving properties for functional calculi of bounded operators. In fact, there are very few results saying that basic features of operator like resolvent estimates or asymptotics of powers, are preserved under a functional calculus or, at least, under a substantial class of admissible functions. It seems, the only relevant and nontrivial result so far was the one by Hirsch [35] saying that complete Bernstein functions preserve the class of sectorial (in general, unbounded) operators.

To put our results into a proper context, we first recall several basic facts stemming from the subordination theory of C_0-semigroups on Banach spaces. There are two basic notions behind the subordination theory: the notion of Bernstein function and that of subordinator.

Recall that a family of positive subprobability Borel measures $(\mu_t)_{t\geq 0}$ on $[0,\infty)$ is said to be said to be a *subordinator* if for all $s,t \geq 0$ one has

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$\mu_{t+s} = \mu_t * \mu_s$, and $\lim_{t\to 0+}\mu_t = \delta_0$ in the w^*-topology of the space of bounded Borel measures on $[0,\infty)$. Given a subordinator $(\mu_t)_{t\geq 0}$ one may define a Bernstein function ψ on $(0,\infty)$ by the formula

$$e^{-t\psi(\lambda)} = \int_0^\infty e^{-s\lambda}\,\mu_t(ds), \tag{1.1}$$

for all $t \geq 0$ and $\lambda > 0$. See [54, Section 5] for more on that. (Alternatively, a positive and smooth function ψ on $(0,\infty)$ is called a *Bernstein function* if $(-1)^n \frac{d^n\psi(t)}{dt^n} \leq 0$ for all $n \in \mathbb{N}$ and $t > 0$.)

If now $(e^{-tA})_{t\geq 0}$ is a bounded C_0-semigroup on a (complex) Banach space X with generator $-A$, and $(\mu_t)_{t\geq 0}$ is a subordinator, then, following intuition provided by (1.1), one can define a new *bounded* C_0-semigroup on X as

$$T(t) := \int_0^\infty e^{-sA}\,\mu_t(ds), \qquad t \geq 0, \tag{1.2}$$

where the (Bochner) integral converges in the strong topology of X. Once again, in view of (1.1), it is natural to consider the generator of $(T(t))_{t\geq 0}$ as a (minus) Bernstein function ψ of A. This appears to be a right choice and can serve as the definition of $\psi(A)$ indeed. There are several other alternative definitions of $\psi(A)$, but all of them lead to the same operator. The operator Bernstein functions have a number of natural properties resembling that of scalar functions. One of these properties is expressed by (1.2) and provides a natural way to construct a bounded semigroup $(e^{-t\psi(A)})_{t\geq 0}$ by means of a given bounded semigroup $(e^{-tA})_{t\geq 0}$ and a subordinator $(\mu_t)_{t\geq 0}$. In this situation, $(e^{-t\psi(A)})_{t\geq 0}$ is called subordinated to $(e^{-tA})_{t\geq 0}$ via a subordinator $(\mu_t)_{t\geq 0}$. The construction of subordination described above goes back to Bochner and Phillips and became a crucial tool in probability theory and functional analysis (and also in engineering), see e.g. [54] and comments to Section 13 there. A classical example of subordination is provided by the semigroup of fractional powers $(e^{-tA^\alpha})_{t\geq 0}, \alpha \in (0,1)$ (corresponding to the Bernstein function $\psi(\lambda) = \lambda^\alpha$). It was studied thoroughly in the 1960s by Balakrishnan, Kato and Yosida. A comprehensive discussion of subordination for C_0-semigroups including many illustrative examples can be found in [54, Section 13].

Apart from a number of issues of a purely probabilistic origin, there are two very natural, operator-theoretical questions in the study of subordination. Namely, whether subordination preserves the classes of holomorphic sectorially bounded holomorphic C_0-semigroups and when it possesses improving properties in the sense that general bounded C_0-semigroups are transformed into holomorphic semigroups. Motivated by a fundamental paper by Carasso and Kato [14] (and also its subsequent developments in [28], [29] and [46]) and answering a problem posed in [39] and [8], we have recently obtained the following result where positive answers to both questions were provided, see [31, Theorems 6.8 and 7.9].

Theorem 1.1. *(a) Let $-A$ be the generator of a bounded C_0-semigroup on X such that A is sectorial of angle $\theta \in [0, \pi/2)$. Then for every Bernstein function ψ the operator $\psi(A)$ is sectorial of the same angle.*
b) Let ψ be a complete Bernstein function and let $\gamma \in (0, \pi/2)$ be fixed. The following assertions are equivalent.

(i) *One has*
$$\psi(\mathbb{C}_+) \subset \overline{\Sigma}_\gamma.$$
(ii) *For each Banach space X and each generator $-A$ of a bounded C_0-semigroup on X, the operator $\psi(A)$ is sectorial of angle γ.*

To create a similar "discrete" framework, let now μ be a probability measure on $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, in other words, $\mu(k) \geq 0, k \geq 0$, and $\sum_{k=0}^\infty \mu(k) = 1$. If T is a power bounded operator on X, then setting $\widehat{\mu}(T) := \sum_{k=0}^\infty \mu(k) T^k$, and denoting $\mu^n := \mu * \cdots * \mu$ the nth convolution power of μ, so that $\mu^n \in \ell_1(\mathbb{Z}_+), n \in \mathbb{N}$, we have

$$\widehat{\mu}(T)^n = \sum_{k=0}^\infty T^k \mu^n(k) = \int_{\mathbb{Z}_+} T^k d\mu^n(k), \tag{1.3}$$

where the last equality is purely formal. Thus, there is a clear analogy with the continuous case, and it is natural to say that $(\widehat{\mu}(T)^n)_{n\geq 0}$ (or $\widehat{\mu}(T)$) is subordinated to $(T^n)_{n\geq 0}$ (or T). However, to simplify our terminology, we will be considering just the powers of $\widehat{\mu}(T)$ defined by means of the power series $\widehat{\mu}(z) = \sum_{k\geq 0} \mu(k) z^k$ called sometimes the generating function of μ. Note that similarly to the case of bounded C_0-semigroups, if T is power bounded, then the operator $\widehat{\mu}(T)$ is power bounded as well. This is precisely the framework of [23], and one may then study finer properties of $\widehat{\mu}(T)$ in terms of the same properties of T. The attempt to set up a discrete subordination similar to the one existing in the setting of C_0-semigroups was also made in [7]. However, the assumptions of [7] seem to be more restrictive than the ones in [23].

The paper [23] is devoted mainly to the study of the improving properties of measures μ, or, equivalently, of their generating functions $\widehat{\mu}$ in the spirit of [14]. To discuss the relevant results from [23] in some more detail, we have to introduce several operator-theoretical notions. A bounded linear operator T on a Banach space X is said to be Ritt if there exists $C \geq 1$ such that

$$\sigma(T) \subset \overline{\mathbb{D}} \qquad \text{and} \qquad \|(\lambda - T)^{-1}\| \leq \frac{C}{|\lambda - 1|}, \quad |\lambda| > 1,$$

where $\overline{\mathbb{D}}$ stands for closure of the open unit disc $\mathbb{D}$. The last two conditions can be equivalently rewritten in the following, formally stronger, form: There exists $\omega \in [0, \pi/2)$ such that

$$\sigma(T) \subset \mathbb{D} \cup \{1\} \qquad \text{and} \qquad \|(\lambda - T)^{-1}\| \leq \frac{C_{\omega'}}{|\lambda - 1|}, \qquad \lambda \in \mathbb{C} \setminus \left(1 - \overline{\Sigma}_{\omega'}\right),$$

for every $\omega' \in (\omega, \pi)$ and an appropriate $C_{\omega'} \geq 1$, where $\Sigma_\omega = \{\lambda \in \mathbb{C} : |\arg \lambda| < \omega\}$ and $\Sigma_0 = (0, \infty)$. We say that T is *of angle* ω in this case.

(In fact, one can put $\omega = \arccos(1/C)$ here, [44].) Moreover, as we prove in Proposition 4.2 below, T is Ritt if and only if there exists $\sigma \geq 1$ such that for every $\sigma' > \sigma$ and some $C_{\sigma'} \geq 1$ one has

$$\|(\lambda - T)^{-1}\| \leq \frac{C_{\sigma'}}{|\lambda - 1|}, \qquad \lambda \in \mathbb{C} \setminus S_{\sigma'},$$

where $S_\sigma := \{z \in \mathbb{D} : |1-z|/(1-|z|) < \sigma\} \cup \{1\}$, is a *Stolz domain.* In this situation T is said to be *of Stolz type* σ. See Section 4 for a thorough discussion of these and related notions. There is a substantial literature devoted to Ritt operators and their various properties ranging from the role in functional calculi to applications in ergodic and probability theories. A sample of it could include [3], [4], [9], [10], [16], [18]–[24], [32], [36], [37], [40]–[44], [47]–[51], [55], and [56]. (Unfortunately, while the topic is vast, there is no survey on Ritt operators yet.) We only note one more characterization of Ritt operators saying that T is Ritt if and only if T is *power bounded*, i.e. $\sup_{n\geq 0} \|T^n\| < \infty$, and $\sup_{n\geq 0} n\|T^n - T^{n+1}\| < \infty$. In fact, Ritt operators can serve as a discrete analogue of generators of (sectorially) bounded holomorphic C_0-semigroups, while power bounded operators correspond to generators of bounded C_0-semigroups. See e.g. [23], [10], [9] or [36] and references therein for comments on that issue.

In analogy with the case of C_0-semigroups considered in [31], one can say that a measure μ is *improving* if for any power bounded operator T on X the operator $\widehat{\mu}(T)$ is Ritt. By means of a general sufficient condition involving the boundary behavior of the generating function $\widehat{\mu}$ in $\mathbb{D}$, Dungey proved in [23] the improving property for several interesting and important probabilities μ.

In this paper, we will put Dungey's results from [23] in a broader context and improve several of them. More generally, in view of the discussion above, it is natural to to try to obtain a discrete counterpart of Theorem 1.1 once the notion of a discrete subordination is adapted. One of the aims of this paper is to prove the results on permanence and improving properties for discrete subordination similar in a sense to Theorem 1.1.

The following statement is one of the main results of this paper. Recall that a closed operator A on X is sectorial with angle of sectoriality $\alpha \in [0, \pi)$ if $\sigma(A) \subset \overline{\Sigma}_\alpha$ and for every $\omega \in (\alpha, \pi)$ there exists $C_\omega > 0$ such that

$$\|\lambda(\lambda - A)^{-1}\| \leq C_\omega, \qquad \lambda \notin \overline{\Sigma}_\omega.$$

Theorem 1.2. *Let*

$$g(\lambda) := \sum_{n=0}^{\infty} c_n \lambda^n, \qquad c_n \geq 0, \quad \sum_{n=0}^{\infty} c_n = 1. \tag{1.4}$$

Then for any Ritt operator T of Stolz type σ on a Banach space X, the operator $g(T)$ is Ritt and of the same Stolz type. Moreover, $g(T)$ is of angle ω, where ω is a sectoriality angle of the Cayley transform $\mathcal{C}(T)$ of T.

This is a discrete counterpart of Theorem 1.1, a). However, the result seems to be stronger than Theorem 1.1, a) (up to a change of frameworks

from the continuous to the discrete one) since no further assumptions are imposed on the sequence $(c_n)_{n\ge 0}$.

Using the terminology from [27], one may call a power series satisfying (1.4) *convex* and formulate a (part of) statement above by saying that *a convex power series of a Ritt operator is Ritt.* This terminology will be used throughout the paper occasionally.

Thus Theorem 1.2 can be considered as a full discrete analogue of Theorem 1.1, a). However additional specific features are present here. While angles of Ritt operators are not, in general, preserved under discrete subordination, we have a good control over them via the Cayley transform. On the other hand, discrete subordination preserves Stolz type of Ritt operators, and probably it is Stolz type that is an adequate substitute of sectoriality angle for sectorial operators in the discrete setting.

Moreover, in this paper, we show that several results of Dungey on improving properties can in fact be derived, more or less directly, from Theorem 1.1, b). This is done by transferring Theorem 1.1, b) to the discrete setting. Moreover, our approach allows us to answer a question left open by Dungey and to provide new interesting examples of improving μ.

Turning to improving properties, we establish an analogue of Theorem 1.1, b) by replacing complete Bernstein functions with Hausdorff functions. This allows us not only to prove alternative proofs for several results from [23] (e.g. Theorems $1.1, 1.2, 1.3$ and Corollary 1.4 there) but also to answer a problem posed in [23, p. 1735] concerning the improving property of the function $f_\epsilon(\lambda) = 1 - \frac{1}{\epsilon}\int_0^\epsilon (1-\lambda)^\alpha \, d\alpha, \epsilon \in (0,1)$. Moreover, our approach allows us to equip the results from [23] with additional geometric properties which are not available via the techniques from [23].

To formulate our second main result, we need a notion of a regular Hausdorff function. We say that $h(\lambda) = c_0 + \sum_{n=1}^\infty c_n\lambda^n, \lambda \in \mathbb{D}$, is a regular Hausdorff function if $c_0 \ge 0$ and there exists a bounded positive Borel measure ν on $[0,1)$ such that

$$c_n = \int_{[0,1)} t^{n-1}\,\nu(dt), \quad n \ge 1, \quad \text{and} \quad c_0 + \int_{[0,1)} \frac{\nu(dt)}{1-t} = 1.$$

Note that if h is as above, then all its Taylor coefficients $c_n, n \ge 0$, are positive and $\sum_{n=0}^\infty c_n = 1$. Thus, h is a generating function of a probability on $\mathbb{Z}_+$ given by $(c_n)_{n\ge 0}$.

Theorem 1.3. *Let h be a non-constant regular Hausdorff function, and let $\gamma \in (0, \pi/2)$ be fixed. The following statements are equivalent.*

(i) *One has*

$$1 - h(\lambda) \subset \overline{\Sigma}_\gamma, \qquad \lambda \in \mathbb{D}.$$

(ii) *For every Banach space X and every power bounded operator T on X the operator $h(T)$ is Ritt of angle γ.*

Let us comment briefly on our techniques and methodology. It is a notable feature of the paper that dealing with bounded operators we apply methods worked out, first of all, to treat unbounded operators. We remark

that the techniques of the present paper is quite different from the techniques from [31]. While [31] put an emphasis on intricate functional calculi arguments, the approach presented here is more direct. It relies on deriving a suitable resolvent representation for a (rotated) Nevanlinna-Pick function of the Cayley transform of T and function-theoretical estimates for certain Nevanlinna-Pick functions. This allows us to obtain results which seem to be more informative than the corresponding statements in [31] (when changing the notions appropriately). As an alternative to functional calculus technique from [31], a similar direct approach to the study of subordination of C_0-semigroups was recently developed in [6]. Note finally that the main results of this paper have recently found interesting applications to the study of convolution operators on $\ell^1(\mathbb{Z})$, see [19] for more details.

2. Notation

For a closed linear operator A on a complex Banach space X we denote by $\operatorname{ran}(A)$, $\ker(A)$ and $\sigma(A)$ the *range*, the *kernel* and the *spectrum* of A, respectively. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$.

The closure of a set S will be denoted by $\overline{S}$, and $f\circ g$ will stand for a composition of functions f and g. For any sets A and B from the complex plane $\mathbb{C}$, we denote $A+B:=\{a+b: a\in A, b\in B\}$ and sometimes write a instead of $\{a\}$.

Finally, we let

$$\mathbb{C}_+=\{\lambda\in\mathbb{C}:\ \operatorname{Re}\lambda>0\},\qquad \mathbb{C}^+=\{\lambda\in\mathbb{C}:\ \operatorname{Im}\lambda>0\},\qquad \mathbb{Z}_+=\mathbb{N}\cup\{0\},$$

and denote

$$\Sigma_0:=(0,\infty),\qquad \Sigma_\beta:=\{\lambda\in\mathbb{C}:\ |\arg\lambda|<\beta\},\ \beta\in(0,\pi],$$

and $\mathbb{D}:=\{\lambda\in\mathbb{C}:|\lambda|<1\}$.

3. Function theory

3.1. **Nevanlinna-Pick and Cayley functions and their mapping properties.** To develop the functional calculi machinery, we have to introduce several function classes and describe their basic properties.

Recall that a function F holomorphic in the upper half-plane $\mathbb{C}^+$ is called Nevanlinna-Pick if $F(\mathbb{C}^+)\subset\overline{\mathbb{C}}_+$. Since we will be interested in functions defined in the right half-plane $\mathbb{C}_+$, we will need a class of rotated Nevanlinna-Pick functions, namely the class of functions holomorphic in $\mathbb{C}_+$ and mapping $\mathbb{C}_+$ into $\overline{\mathbb{C}}_+$. Moreover, the functions from this latter class that are positive on $(0,\infty)$ will play a special role. Thus, eventually, we will work with the class of functions F denoted by $\mathcal{NP}_+$ and described as

$$\mathcal{NP}_+:=\{F \text{ is holomorphic in } \mathbb{C}_+ : F(\mathbb{C}_+)\subset\overline{\mathbb{C}}_+ \text{ and } F((0,\infty))\subset[0,\infty)\}$$

Note that the symmetry principle implies $F(\lambda)=\overline{F(\bar{\lambda})}, \lambda\in\mathbb{C}_+$.

Recall that since the function $f(\lambda) := iF(-i\lambda), \lambda \in \mathbb{C}^+$, is Nevanlinna-Pick, the well-known Herglotz theorem (see e. g. [54, Corollary 6.8]) implies that $f : (0,\infty) \to [0,\infty)$ and

$$f(\lambda) = iF(-i\lambda) = \alpha + a\lambda + \int_{-\infty}^{\infty} \frac{1+\lambda t}{t-\lambda}\,\rho(dt), \qquad \lambda \in \mathbb{C}^+, \tag{3.1}$$

where $\alpha \in \mathbb{R}$, $a \geq 0$, and ρ is a positive finite Borel measure on the real line.

The following theorem contains several properties of $\mathcal{NP}_+$-functions crucial for the sequel.

Theorem 3.1. *Let $F \in \mathcal{NP}_+$. Then the following statements hold.*

(i) *One has*

$$F(\lambda) = a\lambda + \frac{b}{\lambda} + 2\lambda \int_{(0,\infty)} \frac{(1+t^2)\,\rho(dt)}{\lambda^2+t^2}, \qquad \lambda \in \mathbb{C}_+, \tag{3.2}$$

where $a \geq 0$, $b \geq 0$, and ρ is a positive finite Borel measure on $(0,\infty)$.

(ii) *For every $\omega \in [0, \pi/2)$ there exists $c_\omega > 0$ such that*

$$|F(\lambda)| \leq c_\omega \left(|\lambda| + |\lambda|^{-1}\right), \qquad \lambda \in \Sigma_\omega. \tag{3.3}$$

(iii) *For every $\omega \in [0, \pi/2)$,*

$$F(\overline{\Sigma}_\omega \setminus \{0\}) \subset \overline{\Sigma}_\omega. \tag{3.4}$$

(iv) *For all $\beta \in (-\pi/2, \pi/2)$,*

$$\operatorname{Re} F(te^{i\beta}) \geq \cos\beta\, F(t), \qquad t > 0. \tag{3.5}$$

Moreover, for every $c \in (0,1]$,

$$F(t) \geq c\,F(ct), \qquad t > 0, \tag{3.6}$$

and

$$|F(te^{i\beta})| \geq c\cos\beta\, F(ct), \qquad t > 0. \tag{3.7}$$

Proof. The proofs of (i), (iii) and (3.5) can be found e.g. in [15], [30, Corollary 2] (or [52, Theorem 2]), and [12, Theorem 3.4], respectively. The paper [12] contains a unified approach to the proofs of these and similar properties. The property (iii) goes back to [13], and it is a direct consequence of (i).

The property (ii) is an easy consequence of (i) as well, more general estimates can be found in [12].

To prove (3.6), it suffices to note that

$$\frac{t}{t^2+s^2} \geq c\frac{ct}{(ct)^2+s^2}, \qquad t,s > 0, \quad c \in (0,1],$$

hence (3.2) implies (3.6). The statement (3.7) follows from $|F(te^{i\beta})| \geq \operatorname{Re} F(te^{i\beta})$, $t > 0$, and (3.5), (3.6). □

Let $\theta_1, \theta_2 \in (0, \pi]$. We say that $f \in \mathcal{NP}_+(\theta_1, \theta_2)$ if f is holomorphic in Σ_{θ_1} and, moreover

$$f : (0, \infty) \to [0, \infty) \quad \text{and} \quad f(\Sigma_{\theta_1}) \subset \overline{\Sigma}_{\theta_2}.$$

Denote $\mathcal{NP}_+(\theta) := \mathcal{NP}_+(\theta, \theta)$ so that $\mathcal{NP}_+ = \mathcal{NP}_+(\pi/2)$.

Observe that that $f \in \mathcal{NP}_+(\theta_1, \theta_2)$ if and only if

$$F(\lambda) := [f(\lambda^{2\theta_1/\pi})]^{\pi/(2\theta_2)} \in \mathcal{NP}_+. \tag{3.8}$$

The next corollary provides a lower bound for $f \in \mathcal{NP}_+(\theta_1, \theta_2)$ in terms of the restriction of f to $(0, \infty)$.

Corollary 3.2. *Let $f \in \mathcal{NP}_+(\theta_1, \theta_2)$ for some $\theta_1, \theta_2 \in (0, \pi]$. Then for every $\theta \in [0, \theta_1)$,*

$$f(\overline{\Sigma}_\theta \setminus \{0\}) \subset \overline{\Sigma}_{\theta\theta_2/\theta_1}, \tag{3.9}$$

and for all $c \in (0, 1]$ and $\beta \in (-\theta_1, \theta_1)$,

$$|f(te^{i\beta})| \ge c^{2\theta_2/\pi} \cos^{2\theta_2/\pi}\left(\frac{\pi\beta}{2\theta_1}\right) f(c^{2\theta_1/\pi} t), \qquad t > 0. \tag{3.10}$$

Proof. Since, $f(\lambda) = [F(\lambda^{\pi/(2\theta_1)})]^{2\theta_2/\pi}, \lambda \in \Sigma_{\theta_1}$, where F is defined by (3.8), the statement (3.9) follows directly from (3.4). Next, if $t > 0$ and $\beta \in (0, \theta_1)$, then by (3.7) for every $c \in (0, 1]$,

$$\begin{aligned} |f(te^{i\beta})| =& |F(t^{\pi/(2\theta_1)} e^{i\pi\beta/(2\theta_1)})|^{2\theta_2/\pi} \\ \ge& c^{2\theta_2/\pi} \cos^{2\theta_2/\pi}(\pi\beta/(2\theta_1)) [F(ct^{\pi/(2\theta_1)})]^{2\theta_2/\pi} \\ =& c^{2\theta_2/\pi} \cos^{2\theta_2/\pi}(\pi\beta/(2\theta_1)) f(c^{2\theta_1/\pi} t). \end{aligned}$$

□

The *subclass* of $\mathcal{NP}_+$ formed by complete Berntsein functions and denoted by $\mathcal{CBF}$ will also be important in our considerations. Complete Bernstein functions allow a number of equivalent characterizations. The following one, which can serve as the definition of a complete Bernstein function, can be found in [54, Theorem 6.2]. We say that the function $\psi : (0, \infty) \mapsto [0, \infty)$ is *complete Bernstein* if ψ admits an analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$ which is given by

$$\psi(\lambda) = a + b\lambda + \int_{(0,\infty)} \frac{\lambda\, \mu(ds)}{\lambda + s}, \tag{3.11}$$

where $a, b \ge 0$ are non-negative constants and μ is a positive Borel measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} \frac{\mu(ds)}{1 + s} < \infty. \tag{3.12}$$

Given ψ, the triple (a, b, μ) is determined uniquely, and it is called the Stieltjes representation of ψ. The standard examples of complete Bernstein functions include $\lambda^\alpha, \alpha \in [0, 1], \log(1 + \lambda)$ and $\lambda/(\lambda + a), a > 0$.

The class $\mathcal{CBF}$ has a rich structure which is particularly suitable for functional calculi purposes. We will need just a few of them and refer to [54, Sections 6 and 7] for a comprehensive account. In particular, the following elementary properties of $\mathcal{C}BF$ will be useful, see e.g. [54, Theorem 6.2 and Corollary 7.6] for their discussion.

Theorem 3.3. (i) *Let ψ be a holomorphic function in $\mathbb{C} \setminus (-\infty, 0]$. Then $\psi \in \mathcal{CBF}$ if and only if $\psi(\mathbb{C}^+) \subset \overline{\mathbb{C}^+}, \psi((0,\infty)) \subset [0,\infty)$, and there exists $\psi(0+) = \lim_{\lambda\to 0+} \psi(\lambda)$;*
(ii) *Let $\psi, \varphi \in \mathcal{CBF}$. Then $\psi + \varphi, \psi \circ \varphi \in \mathcal{CBF}$.*

The following result allows one to bound the imaginary part of a complete Bernstein function by means of its derivative on the positive half-axis.

Lemma 3.4. *Let $\psi \in \mathcal{CBF}$. Then for all $\beta \in (-\pi, \pi)$ and $t > 0$,*

$$\operatorname{Im} \psi(te^{i\beta}) \le 2t \tan(\beta/2)\, \psi'(t). \tag{3.13}$$

Proof. Let ψ be of the form (3.11) and let $\lambda = te^{i\beta}$. Then observing that

$$\psi'(\lambda) = b + \int_{(0,\infty)} \frac{s\,\mu(ds)}{(\lambda+s)^2},$$

and using the inequality

$$|te^{i\beta} + s|^2 \ge \cos^2(\beta/2)(t+s)^2, \qquad \beta \in (-\pi, \pi),$$

we obtain that

$$\begin{aligned}
\operatorname{Im} \psi(te^{i\beta}) =& bt\sin\beta + \int_{(0,\infty)} \operatorname{Im} \frac{te^{i\beta}}{te^{i\beta}+s}\,\mu(ds) \\
=& t\sin\beta \left(b + \int_{(0,\infty)} \frac{s\,\mu(ds)}{|te^{i\beta}+s|^2} \right) \\
\le& t\sin\beta \left(b + \frac{1}{\cos^2(\beta/2)} \int_{(0,\infty)} \frac{s\,\mu(ds)}{(t+s)^2} \right) \\
\le& \frac{t\sin\beta}{\cos^2(\beta/2)} \left(b + \int_{(0,\infty)} \frac{s\,\mu(ds)}{(t+s)^2} \right) \\
=& 2t\tan(\beta/2)\psi'(t).
\end{aligned}$$

□

Now we introduce a technical condition which will be basic for estimates in subsequent sections. It is a local version of (3.13).

Definition 3.5. Let $a > 0$ and $\theta \in (0, \pi)$. We let $\mathcal{D}_\theta(0, a)$ be the space of holomorphic functions f on Σ_θ such that

a) f is real on $(0, \infty)$ and strictly increasing on $(0, a)$;
b) for every $R > 0$ and every $\beta \in (-\theta, \theta)$ there exist $b = b(\beta, R) \in (0, \min(1, a/R))$ and $m = m(\beta)$ such that

$$|\operatorname{Im} f(te^{i\beta})| \le mtf'(bt) \tag{3.14}$$

for all $t \in (0, R)$.

Observe that by Lemma 3.4, $\mathcal{CBF} \subset \mathcal{D}_\theta(0,a)$ for all $a > 0$ and $\theta \in (0,\pi)$. The next lemma, proved in Appendix A, shows that some functions from $\mathcal{NP}_+$ belong to $\mathcal{D}_{\pi/2}(0,1)$.

Lemma 3.6. *Suppose that*

$$\sum_{n=0}^{\infty} c_n = 1, \qquad c_n \ge 0,\ n \ge 0,$$

and let

$$\mathbf{h}(\lambda) := 1 - \sum_{n=0}^{\infty} c_n \left(\frac{1-\lambda}{1+\lambda}\right)^n, \qquad \lambda \in \mathbb{C}_+.$$

Then $\mathbf{h} \in \mathcal{D}_{\pi/2}(0,1) \cap \mathcal{NP}_+$. *Moreover, the corresponding constants* $b = b(\beta, R)$ *and* $m = m(\beta)$ *from Definition 3.5 are given by*

$$b = \frac{\cos\beta}{1+R^2} \qquad \textit{and} \qquad m = \frac{\pi}{2}.$$

Finally, we will also need the next geometric proposition proved in [31].

Proposition 3.7. [31, Proposition 3.6]. *Assume that for* $\psi \in \mathcal{CBF}$ *there exists* $\omega \in (0, \pi/2)$ *such that*

$$\psi(\mathbb{C}_+) \subset \overline{\Sigma}_\omega. \tag{3.15}$$

Define $\omega_0 \in (\pi/2, \pi)$ *by*

$$|\cos\omega_0| = \frac{\cot\omega}{\cot\omega + 1},$$

and for $\theta \in (\pi/2, \omega_0)$ *define* $\theta_0 \in (0, \pi/2)$ *as*

$$\cot\theta_0 = \frac{\cot\omega - (\cot\omega + 1)|\cos\theta|}{\sin\theta}. \tag{3.16}$$

Then

$$\psi(\overline{\Sigma_\theta}) \subset \overline{\Sigma}_{\theta_0}. \tag{3.17}$$

3.2. A^1_+- **and Hausdorff functions.** Let $\mathrm{A}^1_+(\mathbb{D})$ be the algebra of holomorphic functions f on the unit disc $\mathbb{D}$ that have absolutely summable Taylor coefficients:

$$\mathrm{A}^1_+(\mathbb{D}) := \left\{ f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n,\ \lambda \in \mathbb{D} :\ \sum_{n=0}^{\infty} |c_n| < \infty \right\}.$$

Clearly, if $f \in \mathrm{A}^1_+(\mathbb{D})$ then f is continuous on $\overline{\mathbb{D}}$. Setting

$$\|f\|_{\mathrm{A}^1_+(\mathbb{D})} := \sum_{n=0}^{\infty} |c_n| \qquad \text{if} \quad f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n,$$

one infers that $(\mathrm{A}^1_+(\mathbb{D}), \|f\|_{\mathrm{A}^1_+})$ is a unital commutative Banach algebra with respect to pointwise multiplication.

Consider now functions h given by
(3.18)
$$h(\lambda) = c_0 + \sum_{n=1}^{\infty} c_n\lambda^n, \qquad \text{where } c_0 \ge 0, \quad c_n := \int_{[0,1)} t^{n-1}\,\nu(dt), \quad n \ge 1,$$
and ν is a bounded positive Borel measure on $[0,1)$ such that
$$c_0 + \int_{[0,1)} \frac{\nu(dt)}{1-t} = 1. \tag{3.19}$$
For the purposes of the present paper, the functions h satisfying (3.18) and (3.19) will be called *regular Hausdorff functions* (since the moment sequences (c_n) are often called Hausdorff sequences). We will write $h \sim (c_0, \nu)$ and say that the measure ν is the (Hausdorff) *representing measure* for h.

Observe that if h is defined by (3.18) and (3.19) then
(3.20)
$$h(\lambda) = c_0 + \int_{[0,1)} t^{-1}\left(\sum_{n=1}^{\infty}(t\lambda)^n\right)\nu(dt) = c_0 + \int_{[0,1)} \frac{\lambda\nu(dt)}{1-t\lambda}, \qquad \lambda \in \mathbb{D},$$
and moreover h extends analytically to $\lambda \in \mathbb{C}\setminus[1,\infty)$. By (3.19) and Fatou's lemma,
$$\|h\|_{\mathrm{A}^1_+(\mathbb{D})} = h(1) = \sum_{n=0}^{\infty} c_n = 1. \tag{3.21}$$
The next proposition relates the regular Hausdorff functions to complete Bernstein functions thus connecting the discrete and the continuous settings.

Proposition 3.8. *Let $h \sim (c_0, \nu)$ be a regular Hausdorff function, and let*
$$\psi(\lambda) := 1 - h(1-\lambda), \qquad \lambda \in \mathbb{D}. \tag{3.22}$$
Then ψ extends to a complete Bernstein function of the form $(0, b, \mu)$, where
$$b = \nu(\{0\}), \qquad \mu(dt) = \frac{\nu(ds) - b\delta_0(ds)}{s(1-s)} \qquad (t = (1-s)/s),$$
and δ_0 denotes the Dirac measure at 0.

Conversely, suppose $\psi \in \mathcal{CBF}$ is such that $\psi \sim (0,0,\mu)$. Then
$$h(\lambda) := \psi(1) - \psi(1-\lambda)$$
is a regular Hausdorff function such that $h \sim (0, \nu)$, where
$$\nu(ds) = \frac{t\mu(dt)}{(1+t)^2} \qquad (s = 1/(1+t)).$$

Proof. Let $\nu(ds) = b\delta_0(ds) + \nu_0(ds)$, where $\nu_0(ds)$ is a Borel measure on $(0,1)$. Taking into account (3.18), we have
$$\begin{aligned}\psi(\lambda) =& \int_{[0,1)} \frac{\nu(ds)}{1-s} - \int_{[0,1)} \frac{(1-\lambda)\,\nu(ds)}{1-s+\lambda s} = \int_{[0,1)} \frac{\lambda\,\nu(ds)}{(1-s+\lambda s)(1-s)} \\ =& b\lambda + \int_{(0,1)} \frac{\lambda\,\nu_0(ds)}{((1-s)/s+\lambda)s(1-s)}.\end{aligned}$$

So, passing to the push-forward measure $\mu(dt)$ of $\frac{\nu_0(ds)}{s(1-s)}$ under the map $t:(0,1)\to(0,\infty), t(s)=(1-s)/s$, we obtain that

$$\psi(\lambda)=b\lambda+\int_{(0,\infty)}\frac{\lambda\,\mu(dt)}{t+\lambda},\qquad \mu(dt)=\frac{(t+1)^2\,\nu_0(ds)}{t},$$

and

$$\int_{(0,\infty)}\frac{d\mu(t)}{1+t}=\int_{(0,1)}\frac{\nu(ds)}{1-s}<\infty. \tag{3.23}$$

If $\psi\in\mathcal{CBF}$ and $\psi\sim(0,0,\mu)$, then

$$\begin{aligned}\psi(1)-\psi(1-\lambda)&=\int_{(0,\infty)}\frac{\mu(dt)}{1+t}-\int_{(0,\infty)}\frac{(1-\lambda)\,\mu(dt)}{1-\lambda+t}\\&=\int_{(0,\infty)}\frac{\lambda t\,\mu(dt)}{(1+t-\lambda)(1+t)}\\&=\int_{(0,\infty)}\frac{\lambda t\,\mu(dt)}{(1-\lambda/(1+t))(1+t)^2}.\end{aligned}$$

Passing as above to the push-forward measure $\nu(ds)$ of $\frac{t\mu(dt)}{(1+t)^2}$ under the map $s:(0,\infty)\to(0,1), s(t)=1/(1+t)$, we obtain that

$$h(\lambda)=\psi(1)-\psi(1-\lambda)=\int_{(0,1)}\frac{\lambda\,\nu(ds)}{1-s\lambda},$$

and (3.23) holds. $\square$

To illustrate the second statement in Proposition 3.8 and in view of further applications in Section 7, let us consider the next simple example.

Example 3.9. a) Let $\alpha\in(0,1)$ be fixed. Recall that (see [54, p. 304]) $\psi_\alpha(\lambda):=\lambda^\alpha\in\mathcal{CBF}$ and

$$\psi_\alpha(\lambda)=\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty\frac{\lambda ds}{(\lambda+s)s^{1-\alpha}},\qquad \lambda\in\mathbb{C}\setminus(-\infty,0].$$

Thus, $\psi_\alpha(1)=1$ and $\psi_\alpha\sim(0,0,\mu_\alpha)$, where

$$\mu_\alpha(ds)=\frac{\sin(\pi\alpha)}{\pi}\frac{ds}{s^{1-\alpha}}.$$

Then, by Proposition 3.8, $h_\alpha(\lambda):=1-(1-\lambda)^\alpha, \lambda\in\mathbb{D}$, is a regular Hausdorff function and $h_\alpha\sim(0,\nu_\alpha)$, where

$$\nu_\alpha(dt)=\frac{\sin(\pi\alpha)}{\pi}\frac{\lambda(1-t)^\alpha\,dt}{(1-\lambda t)t^\alpha}.$$

b) For $\psi(\lambda):=\frac{\lambda-1}{\log\lambda}\in\mathcal{CBF}$, we use the representation (see [54, p. 322])

$$\psi(\lambda)=\int_0^\infty\frac{\lambda\,(s+1)\,ds}{(\lambda+s)s(\log^2 s+\pi^2)},\qquad \lambda\in\mathbb{C}\setminus(-\infty,0],$$

so that $\psi \sim (0,0,\mu)$, $\psi(1)=1$, where

$$\mu(ds)=\frac{(s+1)ds}{s(\log^2 s+\pi^2)}.$$

If $h(\lambda)=1-\psi(1-\lambda)=1+\lambda/\log(1-\lambda)$, $\lambda\in\mathbb{D}$, then, by Proposition 3.8 we infer that h is a regular Hausdorff function and $h\sim(0,\nu)$, where

$$\nu(dt)=\frac{dt}{t(\log^2(1/t-1)+\pi^2)}.$$

In Example 3.9, a) and b) one may also write down the Taylor coefficients for h explicitly.

4. Sectorial operators and Ritt operators

In this section we will introduce and discuss sectorial and Ritt operators, the main objects of our studies. Moreover, we recall and study the notion of Stolz domain. This is a geometric notion related to Ritt operators and to some extent matching the notion of sector for sectorial operators. Moreover, we prove several geometric properties of the spectrum of Ritt operators.

Let us first recall that a closed, densely defined operator A is called sectorial with sectoriality angle $\alpha\in[0,\pi)$ if $\sigma(A)\subset\overline{\Sigma}_\alpha$, and for any $\omega\in(\alpha,\pi)$ exists $M(A,\omega)<\infty$ such that

$$\|z(z-A)^{-1}\|\le M(A,\omega), \qquad z\notin\overline{\Sigma}_\omega.$$

The set of the sectorial operators with angle $\alpha\in[0,\pi)$ will be denoted by $\mathrm{Sect}(\alpha)$. Note that $A\in\mathrm{Sect}(\alpha)$ for some $\alpha\in[0,\pi)$ if and only

$$M(A):=\sup_{z>0}\|z(z+A)^{-1}\|<\infty. \tag{4.1}$$

Define also the minimal angle of sectoriality $\alpha(A)$ of a sectorial operator A as

$$\alpha(A):=\inf\{\alpha : A\in\mathrm{Sect}(\alpha)\}.$$

(Note that inf above can never be replaced by min.) In this paper, we will mostly be dealing with bounded sectorial operators, although some operators will a priori be considered as unbounded ones.

As was explained in the introduction, the theory of Ritt operators is well-developed by now and there are many papers treating various aspects of such operators. Being unable to present all important and relevant results, we thus restrict ourselves to discussing only very basic aspects of that theory.

Let us first recall that $T\in\mathcal{L}(X)$ is said to be *Ritt* if $\sigma(T)\subset\overline{\mathbb{D}}$ and there exists $C\ge 1$ such that

$$\|(z-T)^{-1}\|\le\frac{C}{|z-1|}, \qquad z\in\mathbb{C}\setminus\overline{\mathbb{D}}. \tag{4.2}$$

Note that if $C=1$ in (4.2) then necessarily $\sigma(T)=\{1\}$, see [44, p. 154]. There is a direct link between the notion of Ritt operators and the notion of sectorial operators. Recall that $T\in\mathcal{L}(X)$ is Ritt if and only if $\sigma(T)\subset\mathbb{D}\cup\{1\}$ and there is $\omega\in[0,\pi/2)$ such that the semigroup $(e^{-(1-T)z})_{z\in\mathbb{C}}$ is

sectorially bounded in Σ_ω, see e.g. [23, Th. 1.5] and the comments preceding it. This fact allows the following convenient reformulation which we separate for future references.

Theorem 4.1. *An operator $T \in \mathcal{L}(X)$ is Ritt if and only if there exists $\alpha \in [0,\pi/2)$ such that*

$$\sigma(T) \subset (\mathbb{D} \cup \{1\}) \cap \{z \in \mathbb{C} : 1 - z \in \overline{\Sigma}_\alpha\} \quad \text{and} \quad (1-T) \in \mathrm{Sect}(\alpha).$$

Observe that the last condition means that for any $\beta \in (\alpha, \pi/2)$ there exists $C_\beta \geq 1$ such that

$$\|(z-T)^{-1}\| \leq \frac{C_\beta}{|z-1|}, \qquad z \in \mathbb{C} \setminus \left((1-\overline{\Sigma}_\beta) \cap \overline{\mathbb{D}}\right). \tag{4.3}$$

Thus, if (4.3) holds, we will say that *T is a Ritt operator of angle α.*

Note that the Ritt condition (4.2) has a number of implications for the shape of the spectrum of T. To formulate them we need to define several concepts.

For $\sigma \geq 1$ define a *Stolz domain S_σ* by

$$S_\sigma := \{z \in \mathbb{D} : \, |1-z|/(1-|z|) < \sigma\} \cup \{1\}, \tag{4.4}$$

Clearly, $S_\sigma = \{1\}$ if $\sigma = 1$.

To relate Stolz domains to angular sectors, observe that

$$1 - \overline{S}_\sigma \subset \overline{\Sigma}_\omega, \qquad \omega = \arccos(1/\sigma). \tag{4.5}$$

Indeed, let $\sigma > 1$ and $1 \neq z = 1 - \rho e^{i\alpha} \in S_\sigma \subset \mathbb{D}$. Then $\rho < \cos\alpha \leq 1$ and $\rho/(1-|1-\rho e^{i\alpha}|) < \sigma$, or

$$\sigma\,|1-\rho e^{i\alpha}| < \sigma - \rho. \tag{4.6}$$

A direct calculation shows that

$$\rho < \frac{2\sigma}{\sigma^2-1}(\sigma\cos\alpha - 1). \tag{4.7}$$

Therefore, we have, in particular, that

$$\cos\alpha > \frac{1}{\sigma} \qquad \text{and} \qquad (1-S_\sigma) \setminus \{0\} \subset \Sigma_\omega.$$

Remark that $1 - \overline{S}_\sigma$ is not a subset of $\overline{\Sigma}_{\tilde{\omega}}$ for any $\tilde{\omega} < \omega$.

The next result sharpens the definition of Ritt operators in terms of Stolz domains.

Proposition 4.2. *Let T be a Ritt operator satisfying (4.2) for some $C \geq 1$. Then*

$$\sigma(T) \subset \overline{S}_\sigma \quad \text{with } \sigma = C, \tag{4.8}$$

and for any $\delta > \sigma$ there exists C_δ such that

$$\|(1-z)(z-T)^{-1}\| \leq C_\delta, \qquad z \in \mathbb{C} \setminus S_\delta. \tag{4.9}$$

Conversely, if (4.8) and (4.9) hold for some $\sigma \geq 1$, then T is Ritt.

Proof. Note first that if $C = 1$ then $\sigma(T) = \{1\}$ so that (4.9) holds, see [44, p. 154]. Assume now that T is a Ritt operator satisfying (4.2) with $C > 1$. Then, by [44, Proposition 1, Theorem 2 and Corollary],

$$\sigma(T) \subset \Omega(q) := \{z \in \mathbb{D} \cup \{1\} : |z - e^{i\varphi}| \geq q|1 - e^{i\varphi}| \text{ for all } \varphi \in [0, 2\pi)\},$$

where $q = \frac{1}{C}$. Moreover, $\Omega(q)$ is a closed convex set contained in the shifted sector $1 - \overline{\Sigma}_{\arccos q}$, and for any $\delta \in (\arccos q, \pi/2)$,

$$\|(z - T)^{-1}\| \leq \frac{C(\delta)}{|z - 1|}, \quad 1 - z \notin \overline{\Sigma}_\delta, \tag{4.10}$$

where $C(\delta) = \frac{C}{1 - C\cos\delta}$. By Lemma 9.1 (proved in Appendix A), the set $\Omega(q)$ can be described as

$$\Omega(q) \setminus \{1\} = \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{2|1 - z|} \geq q \right\}.$$

Then, since

$$\frac{1 - |z|^2}{2|1 - z|} \leq \frac{1 - |z|}{|1 - z|}, \qquad z \in \mathbb{D},$$

the definition (4.4) of Stolz domain yields

$$\Omega(q) \subset \overline{S}_\sigma, \qquad \sigma = 1/q = C,$$

i.e. (4.8) holds. Then (4.9) follows from (4.5), (4.8) and (4.10).

The converse implication follows from the obvious fact that $S_\sigma \subset \mathbb{D} \cup \{1\}$ for all $\sigma \geq 1$ and a characterization of Ritt operators in terms their sectoriality given in Theorem (4.1) (see e.g. [23, Theorem 1.5]). $\square$

Proposition 4.2 motivates the following definition. An operator $T \in \mathcal{L}(X)$ is said to be *Ritt operator of Stolz type* $\sigma \in [1, \infty)$ if $\sigma(T) \subset \overline{S}_\sigma$ and T satisfies (4.9) for any $\delta > \sigma$.

Remark 4.3. Note that there is an alternative geometric object related to Ritt operators. Namely, define a set B_ω, $\omega \in (0, \pi/2)$, as the interior of the convex hull of 1 and the disc $D_{\sin\omega} := \{z \in \mathbb{C} : |z| < \sin\omega\}$, i.e.

$$\overline{B}_\omega = \overline{\mathrm{co}}\,(D_{\sin\omega} \cup \{1\}).$$

In [42], it is B_ω that is called a Stolz domain, while we use that terminology for S_σ. Note that $B_\omega \subset 1 - \Sigma_\omega$. One can prove that $T \in \mathcal{L}(X)$ is Ritt if and only if there exists $\alpha \in (0, \pi/2)$ such that $\sigma(T) \subset \overline{B}_\alpha$ and for any $\beta \in (\alpha, \pi/2)$ the set $\{(z - 1)(z - T)^{-1} : z \in \mathbb{C} \setminus \overline{B}_\beta\}$ is bounded. See e.g. [42, Definition 2.2] and [42, Lemma 2.1] concerning the above. However, the domains as B_ω appear to be less convenient for the study of the permanence properties of Ritt operators under functional calculi. Thus, we do not discuss them in this paper.

The relevance of Stolz domains is also clear from the statement given below which will be instrumental in the proof of our main assertion.

Proposition 4.4. *Let* $h(\lambda) = \sum_{n=0}^\infty c_n\lambda^n, \lambda \in \overline{\mathbb{D}}, c_n \ge 0,$ *be such that* $\sum_{n=0}^\infty c_n = 1$. *Then for each* $\sigma \ge 1$,

$$h(\overline{S}_\sigma) \subset \overline{S}_\sigma. \tag{4.11}$$

Proof. Since

$$\frac{|1-\lambda^n|}{1-|\lambda^n|} = \frac{|\sum_{k=0}^{n-1}\lambda^k|}{\sum_{k=0}^{n-1}|\lambda|^k}\cdot\frac{|1-\lambda|}{1-|\lambda|} \le \frac{|1-\lambda|}{1-|\lambda|}, \quad \lambda \in \mathbb{D},$$

each of the functions $h_n(\lambda) := \lambda^n$, $n \in \mathbb{Z}_+$, satisfies the relation (4.11). Then, by the convexity $\overline{S}_\sigma$, the inclusion (4.11) holds for any h given by the convex power series $\sum_{n=0}^\infty c_n\lambda^n$. □

4.1. Operator Cayley transform and its relation to Stolz domains.

In this subsection, we will discuss the operator Cayley transform which will our basic tool in reducing considerations in the discrete setting to their half-plane analogues. However, as we already remarked in the introduction, as far as Ritt operators are bounded, the discrete situation has its specifics so that it makes a sense to study it in some more details.

As far as we will be aiming at reducing the arguments for the unit disc to the half-plane setting, the *Cayley transform* $\mathcal{C}$ will clearly play a crucial role. Recall that the Cayley transform is given by

$$\mathcal{C}(\lambda) := \frac{1-\lambda}{1+\lambda}, \quad \lambda \ne -1, \tag{4.12}$$

and that $\mathcal{C}$ maps $\mathbb{D}$ onto $\mathbb{C}_+$ conformally.

The following proposition relates Stolz domains and angular sectors via the Cayley transform, and will be useful in the sequel.

Proposition 4.5. *Let* $\mathcal{C}$ *be the Cayley transform. If* $\sigma \ge 1$ *and* $\omega = \arccos(1/\sigma)$, *then*

$$\mathcal{C}(\overline{S}_\sigma) \subset \overline{\Sigma}_\omega.$$

Proof. Let $\lambda = 1-\rho e^{i\alpha} \in S_\sigma, \lambda \ne 1$. Then

$$\mathcal{C}(\lambda) = \frac{\rho e^{i\alpha}}{2-\rho e^{i\alpha}} = \frac{\rho(2e^{i\alpha}-\rho)}{|2-\rho e^{i\alpha}|^2}.$$

Using (4.6) and (4.7), we obtain

$$\begin{aligned}
\frac{\operatorname{Re}\mathcal{C}(\lambda)}{|\mathcal{C}(\lambda)|} &= \frac{2\cos\alpha-\rho}{|2-\rho e^{i\alpha}|} \\
&\ge \frac{2\cos\alpha-\rho}{1+|1-\rho e^{i\alpha}|} \\
&\ge \frac{2\cos\alpha-\rho}{1+(\sigma-\rho)/\sigma} \\
&= \frac{1}{\sigma}+\frac{2\sigma(\sigma\cos\alpha-1)-(\sigma^2-1)\rho}{\sigma(2\sigma-\rho)} \\
&\ge \frac{1}{\sigma},
\end{aligned}$$

that is $\mathcal{C}(1-\rho e^{i\alpha}) \in \overline{\Sigma}_\omega$, where $\omega = \arccos(1/\sigma)$. □

Now we turn to the operator analogue of $\mathcal{C}$. For $T \in \mathcal{L}(X)$ with $\sigma(T) \subset \overline{\mathbb{D}}$, and $\ker(1+T) = \{0\}$ we define the *Cayley transform* $\mathcal{C}(T)$ as

$$\mathcal{C}(T) := (1-T)(1+T)^{-1}. \tag{4.13}$$

If $\operatorname{ran}(1+T)$ is dense in X then it is straightforward that $\mathcal{C}(T)$ is a closed densely defined operator on X and $\sigma(\mathcal{C}(T)) \subset \overline{\mathbb{C}}_+$. In our considerations, we will always deal with T such that $-1 \notin \sigma(T)$. Thus, in the sequel, $\mathcal{C}(T)$ will always be *bounded.*

Finally note that, by a direct calculation,

$$\mathcal{C}(\mathcal{C}(T)) = T. \tag{4.14}$$

The following simple proposition relates sectoriality of T to that of $\mathcal{C}(T)$.

Proposition 4.6. *Let T be a power-bounded operator on X such that $-1 \notin \sigma(T)$, and let $\sup_{n\ge 0} \|T^n\| := M$. Then for any $\beta \in (\pi/2, \pi)$,*

$$\|(\mathcal{C}(T)-z)^{-1}\| \le \frac{3M(1+\|T\|)}{|z\cos\beta|}, \quad z \notin \overline{\Sigma}_\beta. \tag{4.15}$$

In particular, $\mathcal{C}(T) \in \mathrm{Sect}(\pi/2)$.

Proof. Note first that if $\lambda \notin \sigma(T)$ and $z = \frac{1-\lambda}{1+\lambda}$ then

$$(\mathcal{C}(T)-z)^{-1} = -\frac{(1+\lambda)}{2}(1+T)(T-\lambda)^{-1}. \tag{4.16}$$

Since for $z \notin \overline{\mathbb{C}}_+$ one has $z = \frac{1-\lambda}{1+\lambda}$ with $\lambda = \frac{1-z}{1+z} \notin \overline{\mathbb{D}}$, the identity (4.16) yields

$$\|(\mathcal{C}(T)-z)^{-1}\| \le \frac{|1+\lambda|}{2}\|(1+T)(T-\lambda)^{-1}\|. \tag{4.17}$$

By assumption and the Neumann series expansion we have

$$\|(T-\lambda)^{-1}\| \le \frac{M}{|\lambda|-1}, \quad \lambda \in \mathbb{C}, \quad |\lambda| > 1, \tag{4.18}$$

hence if $z \notin \overline{\mathbb{C}}_+$, then

$$\begin{aligned} |1+\lambda|\|(T-\lambda)^{-1}\| \le& M\frac{|1+\lambda|}{|\lambda|-1} \\ =& \frac{2M}{|1-z|-|1+z|} \\ =& M\frac{|1-z|+|1+z|}{2|\mathrm{Re}\, z|} \\ \le& M\frac{1+|z|}{|z\cos\beta|}. \end{aligned} \tag{4.19}$$

Thus, from (4.17) and (4.19) it follows that if $z \notin \overline{\mathbb{C}}_+$ is such that $|z| < a$, then

$$\|(\mathcal{C}(T)-z)^{-1}\| \le M\frac{(a+1)}{2|z\cos\beta|} \tag{4.20}$$

Next, if $z \notin \overline{\mathbb{C}}_+$ satisfies $|z| \ge a > 1$, then

$$\frac{1+|z|}{|z|} \le \frac{a+1}{a}, \quad \text{and} \quad \frac{|1+\lambda|}{2} = \frac{1}{|1+z|} \le \frac{a}{(a-1)|z|},$$

so using (4.17) and (4.19) and observing that

$$(1+T)(T-\lambda)^{-1} = 1 + (\lambda+1)(T-\lambda)^{-1},$$

we obtain

$$\begin{aligned} \|(\mathcal{C}(T)-z)^{-1}\| \le & \frac{|1+\lambda|}{2}\left(1+|\lambda+1|\|(T-\lambda)^{-1}\|\right) \\ \le & \frac{a}{(a-1)|z|}\left(1+\frac{M(a+1)}{a|\cos\beta|}\right)\frac{M(2a+1)}{(a-1)|\cos\beta|}\cdot\frac{1}{|z|}. \end{aligned} \tag{4.21}$$

Setting finally $a=4$, (4.19) and (4.21) imply (4.15). $\square$

We proceed with revealing an interplay between geometry of the spectrum of Ritt operators and their Cayley transforms.

Proposition 4.7. *If T is a Ritt operator of Stolz type σ, then $\mathcal{C}(T) \in$ Sect(ω) for $\omega = \arccos(1/\sigma)$.*

Proof. Fix $\tilde\sigma > \sigma$. By assumption,

$$\|(T-\lambda)^{-1}\| \le \frac{C_{\tilde\sigma}}{|\lambda-1|}, \quad \lambda \notin S_{\tilde\sigma}. \tag{4.22}$$

If $\tilde\omega = \arccos(1/\tilde\sigma)$ and $z \notin \Sigma_{\tilde\omega}, z \ne 0$, then by Proposition 4.5 there exists $\lambda \notin S_{\tilde\sigma}$ such that $z = (1-\lambda)/(1+\lambda)$. Hence, by (4.16) and (4.22),

$$\|(\mathcal{C}(T)-z)^{-1}\| \le \frac{|1+\lambda|}{2}\|1+T\|\frac{C_{\tilde\sigma}}{|\lambda-1|} = \frac{C_{\tilde\sigma}\|1+T\|}{2|z|},$$

so that $\mathcal{C}(T) \in \mathrm{Sect}(\tilde\omega)$. Since the choice of $\tilde\sigma > \sigma$ is arbitrary, we conclude that $\mathcal{C}(T) \in \mathrm{Sect}(\omega)$. $\square$

5. Functional calculi

5.1. **Holomorphic calculus and operator complete Bernstein functions.** In this subsection we will set up a holomorphic functional calculus of sectorial operators and will state several of its properties important for the sequel. The comprehensive accounts on the extended holomorphic functional calculus can be found in many texts including e.g. [32, Chapter 2] and [41, Section 9], but we still feel that the functional calculi theory is not a part of general background, so we recall its basic features important for our exposition in subsequent subsections.

For $\varphi \in (0,\pi)$, let $\mathcal{O}(\Sigma_\varphi)$ stands for the space of all holomorphic functions on Σ_φ. Define

$$H_0^\infty(\Sigma_\varphi) \quad := \quad \left\{f \in \mathcal{O}(\Sigma_\varphi) : |f(\lambda)| \le C\min(|\lambda|^s, |\lambda|^{-s}) \text{ for some } C, s > 0\right\},$$

and

$$\mathcal{B}(\Sigma_\varphi) := \left\{f \in \mathcal{O}(\Sigma_\varphi) : |f(\lambda)| \le C\max\left(|\lambda|^s, |\lambda|^{-s}\right) \text{ for some } C, s > 0\right\}.$$

Note that $H_0^\infty(\Sigma_\varphi)$ and $\mathcal{B}(\Sigma_\varphi)$ are algebras.

Let $0 \le \alpha < \varphi < \pi$, and let $A \in \mathrm{Sect}(\alpha)$. For $f \in H_0^\infty(\Sigma_\varphi)$ and $\alpha_0 \in (\varphi, \pi)$, define

$$\Phi(f) = f(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha_0}} f(\lambda)(\lambda - A)^{-1}\, d\lambda,$$

where Σ_{α_0} is the downward oriented boundary of Σ_{α_0}. This definition is independent of α_0, and

$$\Phi : H_0^\infty(\Sigma_\varphi) \mapsto \mathcal{L}(X), \qquad \Phi(f) = f(A),$$

is an algebra homomorphism. Let $\tau(\lambda) := \frac{\lambda}{(1+\lambda)^2}$. Assume that A is injective so that $\Phi(\tau) = \tau(A) = A(1+A)^{-2}$ is injective as well.

Since for any $f \in \mathcal{B}(\Sigma_\varphi)$ there is $n \in \mathbb{N}$ such that

$$\tau^n f \in H_0^\infty(\Sigma_\varphi), \tag{5.1}$$

we can define a closed operator $f(A)$ as

$$f(A) = [\tau^n(A)]^{-1}(\tau^n f)(A) = [A(1+A)^{-2}]^{-n}\,(f\cdot\tau^n)(A), \tag{5.2}$$

where

$$(f\cdot\tau^n)(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_{\alpha_0}} \frac{\lambda^n f(\lambda)}{(\lambda+1)^{2n}}(\lambda - A)^{-1}\, d\lambda,$$

according to the above. This definition does not depend on the choice of n as far as (5.1) holds. A mapping

$$\Phi_e : \mathcal{B}(\Sigma_\varphi) \mapsto \mathcal{L}(X), \quad \Phi_e(f) = f(A),$$

is an algebra homomorphism, and it is called *the extended holomorphic functional calculus* for A.

Note that Φ_e formally depends on a choice of φ, but the calculi are consistent with an appropriate identification. Thus we may consider the calculus to be defined on

$$\mathcal{B}[\Sigma_\alpha] := \bigcup_{\alpha<\varphi<\pi} \mathcal{B}(\Sigma_\varphi).$$

It is important to note that in view of Theorem 3.1,(ii) if $\alpha \in [0,\theta_1)$ and $f \in \mathcal{NP}_+(\theta_1,\theta_2)$ then any f can be regularized by τ^2, and so $f(A)$ is defined in the extended holomorphic functional calculus.

The extended holomorphic calculus is governed by usual calculi rules, see [32, Section 2.3, 2.4] for more on that. The following properties of the calculus will be of particular importance for us.

Proposition 5.1. (i) *If f and g belong to $\mathcal{B}[\Sigma_\alpha]$, then the following sum rule and product rule hold:*

$$f(A) + g(A) \subset (f+g)(A), \qquad f(A)g(A) \subset (fg)(A).$$

If $g(A)$ is bounded, then the inclusions above turn into equalities.

(ii) *Let* $f \in \mathcal{B}[\Sigma_{\alpha'}]$ *and* $g \in \mathcal{B}[\Sigma_\alpha]$. *Suppose in addition that* $g(\Sigma_\alpha) \subset \Sigma_{\alpha'}$, $g(A) \in \mathrm{Sect}(\alpha')$, *and* $g(A)$ *is injective. Then* $f \circ g \in \mathcal{B}[\Sigma_\alpha]$, *and the composition rule hold:*

$$(f \circ g)(A) = f(g(A)). \tag{5.3}$$

The importance of sectoriality angles is well-illustrated by the following classical statement on fractional powers of sectorial operators relevant for our subsequent arguments.

Proposition 5.2. *Let* $\alpha \in [0, \pi)$ *and* $q > 0$ *be such that* $q\alpha < \pi$. *Then* $A^q \in \mathrm{Sect}(q\alpha)$. *Moreover, there exists* $M_q(A) > 0$ *such that for every* $\epsilon > 0$

$$\|\lambda(\lambda + (A + \epsilon)^q)^{-1}\| \le M_q(A), \qquad \lambda \in (0, \infty). \tag{5.4}$$

For a proof of the first part of the proposition see e.g. [32, Proposition 3.1.2] or [8, Corollary 3.10]. The estimate (5.4) is a direct consequence of [32, Corollary 3.1.3] and [32, Proposition 2.1.2, f)].

Complete Bernstein functions introduced in Subsection 3.1 fall into the scope of the extended holomorphic functional calculus. Moreover, such functions can be defined for any sectorial operator regardless of its angle of sectoriality. Indeed, every complete Bernstein function extends holomorphically to $\mathbb{C} \setminus (-\infty, 0]$, and (3.11) implies that it has a sublinear growth in any sector Σ_α, $\alpha \in [0, \pi)$. Identifying a complete Bernstein function with its holomorphic extension to $\mathbb{C} \setminus (-\infty, 0]$, we infer that it belongs to the extended holomorphic functional calculus for any sectorial operator A. The definition (5.2) applies in this case with $n = 2$. Moreover, the following operator analogue of (3.11) holds, see e.g. [5, Theorem 3.12 and Section 3] for its discussion and proof (as well as for more details on the holomorphic functional calculus of complete Bernstein functions). Another approach to operator complete Bernstein functions can be found in [8] and [53].

Theorem 5.3. *Let a complete Bernstein function* ψ *be given by its Stieltjes representation* (a, b, μ) *(see* (3.11)*). Then for every* x *from the domain* $\mathrm{dom}(A)$ *of* A,

$$\psi(A)x = a + bAx + \int_{(0,\infty)} A(A + s)^{-1}x\, \mu(ds). \tag{5.5}$$

Moreover, $\mathrm{dom}(A)$ *is a core for* $\psi(A)$.

Note that a complete Bernstein function ψ of A can also defined in the framework of other calculi, e.g. Hille-Phillips functional calculus, where the assumption $\ker(A) = \{0\}$ is not, in fact, necessary. However, we will not need this fact in the sequel.

5.2. A^1_+**-calculus.** Now we turn to a discussion of another calculus tailored to deal with power bounded operators rather than sectorial ones. If T is a power bounded operator on X, then we can define a $\mathrm{A}^1_+(\mathbb{D})$-functional calculus for T which does not require holomorphicity of functions on $\sigma(T)$ as in the case of the holomorphic functional calculus from the previous subsection. The notion of the Hausdorff function will be crucial in this context. We

will show that the notion is just another face of the notion of the complete Bernstein function explained in the previous subsection.

Since $\mathrm{A}^1_+(\mathbb{D})$ is a convolution Banach algebra, there is a very natural way to define a function from $\mathrm{A}^1_+(\mathbb{D})$ of T. Namely, if $f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n \in \mathrm{A}^1_+(\mathbb{D})$ then we set

$$f(T) = \sum_{n=0}^{\infty} c_n T^n.$$

The mapping

$$\Phi : \mathrm{A}^1_+(\mathbb{D}) \mapsto \mathcal{L}(X), \qquad \Phi(f) = f(T),$$

is a continuous homomorphism of Banach algebras satisfying

$$\|\Phi(f)\| \le \left(\sup_{n\ge 0} \|T^n\|\right) \|f\|_{\mathrm{A}^1_+(\mathbb{D})}.$$

It is called *the* $\mathrm{A}^1_+(\mathbb{D})$*-calculus for* T.

In what follows, we will need a spectral mapping theorem for $\mathrm{A}^1_+(\mathbb{D})$-calculus. This result can be found e.g. in [23, Theorem 2.1].

Proposition 5.4. *Let* $f \in \mathrm{A}^1_+(\mathbb{D})$ *and let* T *be a power-bounded operator on* X*. Then*

$$\sigma(f(T)) = f(\sigma(T)). \tag{5.6}$$

Remark 5.5. Recall that if T_1 and T_2 are commuting bounded operators on X then

$$\operatorname{dist}(\sigma(T_1), \sigma(T_2)) \le \|T_1 - T_2\|, \tag{5.7}$$

where $\operatorname{dist}(\sigma(T_1), \sigma(T_2))$ stands for the Hausdorff distance between $\sigma(T_1)$ and $\sigma(T_2)$. (See e.g. [38, Theorem IV.3.6].) The proof of Proposition 5.4 given in [23] is based on this result, and the result will also be useful in the sequel.

It is important to observe that since $\mathrm{A}^1_+(\mathbb{D})$ includes regular Hausdorff functions, a Hausdorff function h of a power bounded operator T is well-defined in the $\mathrm{A}^1_+(\mathbb{D})$-calculus. Moreover, for $h \sim (c_0, \nu)$ one can prove the operator counterpart of (3.20):

$$h(T) = c_0 + \int_{[0,1)} T(1 - tT)^{-1}\, \nu(dt).$$

As we will not use this formula in the following, its proof is omitted.

Since we will use the two functional calculi, namely the extended holomorphic functional calculus and $\mathrm{A}^1_+(\mathbb{D})$-calculus, a natural question is whether these calculi are consistent. To clarify this issue, note that if T is power bounded, then $1 - T$ is sectorial and by [33, Proposition 3.2] the $\mathrm{A}^1_+(\mathbb{D})$ -calculus agrees with the holomorphic functional calculus for sectorial operators in a sense that for appropriate holomorphic f one has $f(A) = g(T)$ where $A := 1 - T$ has dense range and $g(\lambda) = f(1 - \lambda)$.

Moreover, if $h \sim (c_0, \nu)$ is a regular Hausdorff function, then $\psi(\lambda) := 1 - h(1 - \lambda) \in \mathcal{CBF}$ by Proposition 3.8. Thus, $\psi(A)$ is defined by the

extended holomorphic functional calculus. On the other hand, $h(T)$ can be defined by the $\mathrm{A}^1_+(\mathbb{D})$-calculus. In view of the next result proved in [33, Proposition 3.2] and formulated for a future reference, the two calculi are consistent and lead to the same operator.

Lemma 5.6. *Let h be a regular Hausdorff function and let $\psi(\lambda) = 1-h(1-\lambda)$ be the corresponding complete Bernstein function given by Proposition 3.8. If T is a power bounded operator on X such that $\overline{\operatorname{ran}}\,(1-T) = X$, then*

$$1 - h(T) = \psi(A), \qquad A := 1 - T,$$

where $h(T)$ is defined by means of the $\mathrm{A}^1_+(\mathbb{D})$-calculus and $\psi(A)$ is defined by the extended holomorphic functional calculus.

Remark 5.7. Recall that, by the mean ergodic theorem, if T is a power bounded operator on X and $\overline{\operatorname{ran}}\,(1-T) = X$, then $\ker(1-T) = \{0\}$.

Remark 5.8. Using the approach of Subsection 5.1, one may also define the extended $\mathrm{A}^1_+(\mathbb{D})$-calculus. In this way, the extended $\mathrm{A}^1_+(\mathbb{D})$-calculus for T comprises more general Hausdorff functions of T. However our arguments will not require this generalization.

6. A^1_+-functions of Ritt operators

We now turn to deriving estimates for $(z+f)^{-1}(A)$ where $f = h\circ\mathcal{C} \in \mathcal{NP}_+$ is a convex power series h of the Cayley transform $\mathcal{C}$ and A is a sectorial and *bounded* operator on X. We start with obtaining an integral representation for the "resolvent" function $(z+f)^{-1}$. This representation will lead to a similar representation for $(z+f)^{-1}(A)$, and eventually to the ("uniform") sectoriality of $f(A)$. Finally, if T is Ritt and $A = \mathcal{C}(T)$, then the sectoriality of $f(A)$ with an appropriate angle will imply that $h(T)$ is Ritt.

It is also important to note that our arguments depend essentially on a convergence of certain approximations of Ritt and power bounded operators. Thus all constants in the resolvent bounds given below have been written explicitly so to reveal their uniformity with respect to approximation and to keep control over the convergence issues.

Lemma 6.1. *Let $f \in \mathcal{NP}_+(\theta_1,\theta_2)$. If*

$$\alpha \in (0,\theta_1), \quad q \in (\pi/\theta_1, \pi/\alpha) \quad \text{and} \quad \gamma \in \left(0, \pi\left(1 - \frac{\theta_2}{q\theta_1}\right)\right),$$

then for every $R > 0$,

$$(z+f(\lambda))^{-1} = \frac{q}{\pi}\int_0^{R^{1/q}} \frac{\operatorname{Im} f(te^{i\pi/q})\, t^{q-1}\, dt}{(z+f(te^{i\pi/q}))(z+f(te^{-i\pi/q}))(t^q+\lambda^q)} + \frac{1}{2\pi i}\int_{|\xi|=R} \frac{d\xi}{(z+f(\xi^{1/q}))(\xi-\lambda^q)}, \tag{6.1}$$

for all $\lambda \in \Sigma_\alpha$, $|\lambda| < R^{1/q}$, and $z \in \Sigma_\gamma$.

Proof. Let $\alpha \in (0,\theta_1)$, $q \in (\pi/\theta_1, \pi/\alpha)$ and $R>0$ be fixed. By Corollary 3.2, for every $\beta \in (q\alpha,\pi)$ and all nonzero $\lambda \in \Sigma_\alpha$ and $\xi \in \partial\Sigma_\beta$,

$$f(\lambda) \in \overline{\Sigma}_{\alpha\theta_2/\theta_1} \quad \text{and} \quad f(\xi^{1/q}) \in \overline{\Sigma}_{\beta\theta_2/(q\theta_1)}. \tag{6.2}$$

Note also that if $\lambda \in \Sigma_\alpha$, and $z \in \Sigma_\gamma$, where $\gamma \in (0, \pi(1-\theta_2/(q\theta_1)))$, then

$$\alpha\theta_2/\theta_1 + \gamma < \pi \quad \text{and} \quad \gamma + \beta\theta_2/(q\theta_1) < \pi.$$

Now, by Cauchy's theorem, for every $R > |\lambda|^q$,

$$(z+f(\lambda))^{-1} = \frac{1}{2\pi i}\int_{\partial\Sigma_\beta(R)} \frac{d\xi}{(z+f(\xi^{1/q}))(\xi-\lambda^q)}, \tag{6.3}$$

where $\Sigma_\beta(R) := \Sigma_\beta \cap \{z \in \mathbb{C} : |z| = R\}$. Deforming the contour $\Sigma_\beta(R)$ in (6.3) to the negative semi-axis, we obtain

$$\begin{aligned}
(z+f(\lambda))^{-1} =& \frac{1}{2\pi i}\int_{|\xi|=R} \frac{d\xi}{(z+f(\xi^{1/q}))(\xi-\lambda^q)} \\
&- \frac{1}{2\pi i}\int_0^R \frac{ds}{(z+f(s^{1/q}e^{i\pi/q}))(s+\lambda^q)} \\
&+ \frac{1}{2\pi i}\int_0^R \frac{ds}{(z+f(s^{1/q}e^{-i\pi/q}))(s+\lambda^q)} \\
=& \frac{1}{2\pi i}\int_{|\xi|=R} \frac{d\xi}{(z+f(\xi^{1/q}))(\xi-\lambda^q)} \\
&+ \frac{1}{\pi}\int_0^R \frac{\operatorname{Im} f(s^{1/q}e^{i\pi/q})\, ds}{(z+f(s^{1/q}e^{i\pi/q}))(z+f(s^{1/q}e^{-i\pi/q}))(s+\lambda^q)},
\end{aligned}$$

and (6.1) follows. □

The above lemma is in fact the heart of our strategy. Using the representation (6.3) containing $f(\xi^{1/q})$ rather than $f(\xi)$ we are able to deform the integration contour to the negative half-axis so that to pass to the formula (6.1) containing $\operatorname{Im} f(te^{i\pi/q})$. In turn, this latter term $\operatorname{Im} f(te^{i\pi/q})$, for certain $f \in \mathcal{NP}_+$, allows useful estimates, e.g the one given by Lemma 9.4 from Appendix A. Lemma 9.4 provides a way to cancel singularity of the integrand in (6.1) at $t=0$ and thus helps to show that the integral (6.1) converges absolutely. This is the key point in obtaining resolvent bounds in Theorem 6.3 below.

Next, using the preceding result and Theorem 3.1, we prove the sectoriality of $f(A)$ if $f \in \mathcal{NP}_+(\theta_1,\theta_2) \cap \mathcal{D}_{\theta_1}(0,a)$ for some $a>0$. The proof is based on the integral representation for the resolvent of $f(A)$. The representation yields the sectoriality of $f(A)$ by means of Theorem 3.1 and bounds on $\operatorname{Im} f$ contained in the definition of $\mathcal{D}_{\theta_1}(0,a)$. Moreover, we give a very explicit bound for the resolvent of $f(A)$. It is important for subsequent approximation arguments where a certain uniformity of resolvent estimates is required.

On this way the following lemma from [2, Appendix B, Proposition B5] will also be crucial.

Lemma 6.2. *Let A be a closed densely defined operator on X, and U be a connected open subset of $\mathbb{C}$. Suppose that $U\cap\rho(A)$ is nonempty and that there is a holomorphic function $F : U \to \mathcal{L}(X)$ such that $\{z \in U \cap \rho(A) : F(z) = (z - A)^{-1}\}$ has a limit point in U. Then $U \subset \rho(A)$ and $F(z) = (z - A)^{-1}$ for all $z \in U$.*

Theorem 6.3. *Let $f \in \mathcal{NP}_+(\theta_1, \theta_2) \cap \mathcal{D}_{\theta_1}(0, a)$ for some $a > 0$. Let $A \in \mathcal{L}(X)$ be such that $A \in \mathrm{Sect}(\alpha)$, $\alpha \in [0, \theta_1)$, and $\ker A = \{0\}$. Then*

$$f(A) \in \mathrm{Sect}(\tilde{\alpha}), \quad \tilde{\alpha} = \frac{\theta_2}{\theta_1}\cdot \alpha.$$

Moreover, for all $q \in (\pi/\theta_1, \pi/\alpha)$ and $\gamma \in \left(0, \pi\left(1 - \frac{\theta_2}{q\theta_1}\right)\right)$, one has

$$\|(z + f(A))^{-1}\| \le \frac{c_{q,\gamma}}{|z|}, \qquad z \in \Sigma_\gamma, \tag{6.4}$$

where

$$c_{q,\gamma} = \frac{qM_q(A)m(\pi/q)}{Cb\pi\cos^2((\pi/q + \gamma)/2)} + \frac{2}{\cos((\pi/q + \gamma)/2)}, \tag{6.5}$$

with $M_q(A)$ given by (5.4), $b = b(\pi/q, 2\|A\|)$ and $m = m(\pi/q)$ corresponding to f by the definition of $\mathcal{D}_{\theta_1}(0, a)$, and finally $C = b^{\theta_2/\theta_1}[\cos(\pi^2/(2\theta_1 q))]^{2\theta_2/\pi}$.

Proof. Let $q \in (\pi/\theta_1, \pi/\alpha)$ and $\gamma \in \left(0, \pi\left(1 - \frac{\theta_2}{q\theta_1}\right)\right)$ be fixed, and set $R = 2^q\|A^q\|$. Having in mind (6.1), let us set formally

$$\begin{aligned} R_q(z; f, A) \quad := \quad & \frac{q}{\pi}\int_0^{R^{1/q}} \frac{\mathrm{Im}\, f(te^{i\pi/q})\, t^{q-1}(A^q + t^q)^{-1}\, dt}{(z + f(te^{i\pi/q}))(z + f(te^{-i\pi/q}))} \\ - \quad & \frac{1}{2\pi i}\int_{|\xi|=R} \frac{(\xi - A^q)^{-1}\, d\xi}{z + f(\xi^{1/q})}, \qquad z \in \Sigma_\gamma. \end{aligned} \tag{6.6}$$

We first prove that $R_q(\cdot; f, A) : \Sigma_\gamma \mapsto \mathcal{L}(X)$ is a well-defined holomorphic function and derive a bound for $\|zR_q(z; f, A)\|$ when $z \in \Sigma_\gamma$.

We consider each of the two terms in (6.6) separately. To estimate the first term, note that by Corollary 3.2

$$f(te^{i\beta}) \in \overline{\Sigma}_{\pi/q}, \quad t > 0, \quad |\beta| \le \pi/q. \tag{6.7}$$

Then, by Lemma 9.2 (from Appendix A) and Corollary 3.2, for all $c \in (0, 1]$ and all $z \in \Sigma_\gamma$,

(6.8)

$$\begin{aligned} |(z + f(te^{i\pi/q}))(z + f(te^{-i\pi/q}))| &\ge \cos^2((\pi/q + \gamma)/2)\, \left(|z| + |f(te^{i\pi/q})|\right)^2 \\ &\ge \cos^2((\pi/q + \gamma)/2)\, (|z| + C\, f(\delta t))^2, \end{aligned}$$

where

$$C = c^{2\theta_2/\pi}[\cos(\pi^2/(2q\theta_1))]^{2\theta_2/\pi} \quad \text{and} \quad \delta = \delta(c) := c^{2\theta_1/\pi}.$$

Now, let $b = b(\pi/q, R^{1/q})$ and $m = m(\pi/q)$ be given for f by the definition of $\mathcal{D}_{\theta_1}(0,a)$. Put

$$c := b^{\pi/(2\theta_1)} \in (0,1], \tag{6.9}$$

so that

$$b = \delta = c^{2\theta_1/\pi} \quad \text{and} \quad C = b^{\theta_2/\theta_1}[\cos(\pi^2/(2q\theta_1))]^{2\theta_2/\pi}.$$

According to (3.14), we have

$$|\operatorname{Im} f(te^{i\pi/q})| \le m(\pi/q) t f'(bt), \quad t \in (0, R^{1/q}).$$

Furthermore, by Proposition 5.2, A^q is sectorial, and by (5.4),

$$\|(A^q + t^q)^{-1}\| \le \frac{M_q(A)}{t^q}, \qquad t > 0.$$

Taking the above bounds into account, we then proceed as follows:

$$\begin{aligned} &\int_0^{R^{1/q}} \frac{|\operatorname{Im} f(te^{i\pi/q})|\, \|(A^q+t^q)^{-1}\| t^{q-1}\, dt}{|(z+f(te^{i\pi/q}))(z+f(te^{-i\pi/q}))|} \\ \le\ & M_q \int_0^{R^{1/q}} \frac{|\operatorname{Im} f(te^{i\pi/q})|\, dt}{|(z+f(te^{i\pi/q}))(z+f(te^{-i\pi/q}))| t} \\ \le\ & \frac{M_q(A) m(\pi/q)}{\cos^2((\pi/q+\gamma)/2)} \int_0^{R^{1/q}} \frac{f'(bt)\, dt}{(|z| + C\, f(bt))^2} \\ \le\ & \frac{M_q(A) m(\pi/q)}{Cb\cos^2((\pi/q+\gamma)/2)} \cdot \frac{1}{|z|}. \end{aligned} \tag{6.10}$$

Next, we estimate the second term in (6.6). Note that

$$\|(\xi - A^q)^{-1}\| \le \frac{1}{|\xi| - \|A^q\|} \le \frac{2}{|\xi|}, \qquad |\xi| \ge 2\|A^q\|.$$

Using once again Lemma 9.2 and taking into account (6.7), we infer that

$$\begin{aligned} \frac{1}{2\pi}\int_{|\xi|=R} \frac{\|(\xi - A^q)^{-1}\|\, |d\xi|}{|z + f(\xi^{1/q})|} &\le \frac{1}{\pi R \cos((\pi/q+\gamma)/2)} \int_{|\xi|=R} \frac{|d\xi|}{|z| + |f(\xi^{1/q})|} \\ &\le \frac{2}{\cos((\pi/q+\gamma)/2)} \cdot \frac{1}{|z|}, \qquad z \in \Sigma_\gamma. \end{aligned} \tag{6.11}$$

Finally, since the integrals in (6.6) converge absolutely, the operator-valued function $R_q(\cdot; f, A) : \Sigma_\gamma \mapsto \mathcal{L}(X)$ is holomorphic by a standard application of the Morera theorem.

Thus, due to (6.6), (6.10) and (6.11), $R_q(\cdot; f, A) : \Sigma_\gamma \mapsto \mathcal{L}(X)$ is holomorphic for every $\gamma \in (0, \pi(1-q^{-1}))$. Moreover,

$$\|R_q(z; f, A)\| \le \frac{c_{q,\gamma}}{|z|}, \qquad z \in \Sigma_\gamma, \tag{6.12}$$

where $c_{q,\gamma}$ is defined by (6.5). (Note that $c_{q,\gamma}$ depends only on q, γ, ω, $\|A\|$ and $M_q(A)$.)

Next we show that if $z\in\Sigma_\gamma$, then $R_q(\cdot;f,A)$ coincides with $(z+f(A))^{-1}$, and as a consequence that (6.4) holds. From Lemma 6.2 it follows that it suffices to prove (6.6) only for $z>0$.

So, let $z>0$ be fixed. Since A has trivial kernel, and the function $(z+\cdot)^{-1}$ is bounded on $\mathbb{C}_+$ for every $z>0$, $(z+f)^{-1}(A)$ is defined in the extended holomorphic calculus via (5.2) with $n=1$. Moreover, using Lemma 6.1, for all $\tilde\omega\in(\alpha,\omega)$ and $z>0$,

$$\begin{aligned}
&\left(\frac{\lambda}{(\lambda+1)^2(z+f)}\right)(A)=\frac{1}{2\pi i}\int_{\partial\Sigma_{\tilde\omega}}\frac{\lambda}{(\lambda+1)^2}\frac{(\lambda-A)^{-1}}{(z+f(\lambda))}\,d\lambda\\
=&\frac{q}{2\pi^2 i}\int_0^{R^{1/q}}\frac{\operatorname{Im} f(te^{i\pi/q})\,t^{q-1}}{(z+f(te^{i\pi/q}))(z+f(te^{-i\pi/q}))}\int_{\partial\Sigma_{\tilde\omega}}\frac{\lambda(\lambda-A)^{-1}}{(\lambda+1)^2(\lambda^q+t^q)}\,d\lambda dt\\
+&\frac{1}{(2\pi i)^2}\int_{|\xi|=R}\frac{1}{(z+f(\xi^{1/q}))}\int_{\partial\Sigma_{\tilde\omega}}\frac{\lambda}{(\lambda+1)^2}\frac{(\lambda-A)^{-1}}{(\xi-\lambda^q)}\,d\lambda\,d\xi\\
=&\frac{q}{\pi}\int_0^{R^{1/q}}\frac{\operatorname{Im} f(te^{i\pi/q})\,t^{q-1}}{(z+f(te^{i\pi/q}))(z+f(te^{-i\pi/q}))}A(A+1)^{-2}(A^q+t^q)^{-1}\,dt\\
-&\frac{1}{2\pi i}\int_{|\xi|=R}\frac{1}{(z+f(\xi^{1/q}))}A(A+1)^{-2}(\xi-A^q)^{-1}\,d\xi\\
=&A(A+1)^{-2}R_q(z;f,A).
\end{aligned}$$

Hence, by (5.2),

$$(z+f)^{-1}(A)=R_q(z;f,A)$$

for all $z>0$. Now the product rule for the extended holomorphic functional calculus (Proposition 5.1, (i)) yields

$$\begin{aligned}
R_q(z;f,A)(z+f(A)) &\subset (z+f(A))R_q(z;f,A)\\
&= (z+f)(A)(z+f)^{-1}(A)=1.
\end{aligned}$$

In other words, we have $R_q(z;f,A)=(z+f(A))^{-1}$ for each $z>0$, and then for each $z\in\Sigma_\gamma$. Hence, in particular, $(z+f(A))^{-1}$ satisfies (6.12).

Thus, from (6.6) and (6.12) it follows that $f(A)\in\operatorname{Sect}(\pi-\gamma)$, where $\pi-\gamma\in(\pi/q,\pi)$. Since q can be made arbitrarily close to π/α, so that γ is arbitrarily close to $\pi-\alpha$, we conclude that $f(A)\in\operatorname{Sect}(\alpha)$. □

Next we obtain a corollary of Theorem 6.3 for certain functions from $\mathcal{NP}_+$ arising in the study of convex power series of Ritt operators. Let

$$h(\lambda):=\sum_{n=0}^\infty c_n\lambda^n,\qquad \lambda\in\mathbb{D},\quad c_n\ge 0,\quad \sum_{n=0}^\infty c_n=1, \tag{6.13}$$

and

$$\mathbf{h}(\lambda):=1-h\left(\frac{1-\lambda}{1+\lambda}\right)=1-\sum_{n=0}^\infty c_n\left(\frac{1-\lambda}{1+\lambda}\right)^n,\qquad \lambda\in\overline{\mathbb{C}}_+. \tag{6.14}$$

The next result is a direct corollary of Theorem 6.3.

Theorem 6.4. *Let* $\mathbf{h}$ *be given by (6.14), and let* $A \in \mathcal{L}(X)$ *be such that* $A \in \mathrm{Sect}(\alpha)$, $\alpha \in [0, \pi/2)$, *and* $\ker A = \{0\}$. *Then* $\mathbf{h}(A)$ *is defined in the extended holomorphic functional calculus, and* $\mathbf{h}(A) \in \mathrm{Sect}(\alpha)$. *Moreover, for all* $q \in (2, \pi/\alpha)$ *and* $\gamma \in \left(0, \pi\left(1 - \frac{1}{q}\right)\right)$, *one has*

$$\|(z + \mathbf{h}(A))^{-1}\| \le \frac{c_{q,\gamma}}{|z|}, \qquad z \in \Sigma_\gamma, \tag{6.15}$$

where

$$c_{q,\gamma} = \frac{qM_q(A)}{2b^2 \cos(\pi/q)\cos^2(\pi/q + \gamma)/2} + \frac{2}{\cos(\pi/q + \gamma)/2}, \tag{6.16}$$

with $b = b_{q,\|A\|} = \frac{\cos(\pi/q)}{1+4\|A\|^2}$, *and* $M_q(A)$ *given by* (5.4).

Proof. By Lemma 3.6, $\mathbf{h}$ belongs to $\mathcal{D}_{\pi/2}(0,1) \cap \mathcal{NP}_+$. Moreover $\mathbf{h}$ satisfies Definition 3.5 of $\mathcal{D}_{\pi/2}(0,1)$ with

$$b = b(\beta, R) = \frac{\cos\beta}{1 + R^2} \in (0, \min\{1, 1/(2R)\})$$

and $m(\beta) = \pi/2$. Therefore, by Theorem 6.3 (with $\theta_1 = \theta_2 = \pi/2$), we get (6.15) and (6.16). $\square$

We proceed with obtaining a counterpart of the preceding result for complete Bernstein functions of sectorial operators. It will be needed in the next section in the study of improving properties of Hausdorff functions. As above, the explicit constants will be given since they will be crucial for the sequel.

Theorem 6.5. *Suppose* $\psi \in \mathcal{CBF}$ *and* (3.15) *holds for some* $\omega \in (0, \pi/2)$. *Let* $A \in \mathcal{L}(X)$ *be such that* $A \in \mathrm{Sect}(\pi/2)$ *and* $\ker A = \{0\}$. *Then* $\psi(A) \in \mathrm{Sect}(\omega)$. *If the numbers* ω_0, θ *and* θ_0 *are as in Proposition 3.7, then for all* $q \in (\pi/\theta, 2)$ *and* $\gamma \in \left(0, \pi\left(1 - \frac{\theta_0}{q\theta}\right)\right)$, *one has*

$$\|(z + \psi(A))^{-1}\| \le \frac{c_{q,\gamma}}{|z|}, \qquad z \in \Sigma_\gamma, \tag{6.17}$$

where

$$c_{q,\gamma} := \frac{2qM_q(A)\tan(\pi/(2q))}{C\pi\cos^2(\pi/q + \gamma)/2} + \frac{2}{\cos(\pi/q + \gamma)/2}. \tag{6.18}$$

with $C := [\cos(\pi^2/(2\theta q))]^{2\theta_0/\pi}$ *and* $M_q(A)$ *given by* (5.4).

Proof. Fix $\theta \in (\pi/2, \omega_0)$. Let ω_0 and θ_0 be defined as in Proposition 3.7. Then

$$\psi \in \mathcal{NP}_+(\theta, \theta_0).$$

Moreover, from Lemma 3.4 it follows that $\psi \in \mathcal{D}_\theta(0, a)$ for each $a > 0$ with

$$m(\beta) = 2\tan(\beta/2) \quad \text{and} \quad b = b(\beta, R) = 1, \quad \beta \in (0, \pi/2), \quad R > 0.$$

Then, by Theorem 6.3, for $\theta_1 = \theta$, $\theta_2 = \theta_0$ and $\alpha = \pi/2$, $\psi(A)$ is defined in the extended holomorphic functional calculus, and

$$\psi(A) \in \mathrm{Sect}(\omega(\theta)), \qquad \omega(\theta) = \frac{\theta_0}{\theta} \cdot \frac{\pi}{2}. \tag{6.19}$$

Moreover, (6.17) holds with the constant given by (6.18).

Finally, from (3.16) it follows that

$$\lim_{\theta \to \pi/2} \theta_0(\theta) = \omega,$$

hence, according to (6.19), $\lim_{\theta\to\pi/2} \omega(\theta) = \omega$. So, by the definition of a sectorial operator, $\psi(A) \in \mathrm{Sect}(\omega)$. □

We turn to the proof of the main result of this paper on convex power series of Ritt operators. To this aim, it will be convenient to separate the next simple technical statement as a lemma.

Lemma 6.6. *Let g_ϵ, $\epsilon \geq 0$, be given by*

$$g_\epsilon(\lambda) = \frac{(2-\epsilon)\lambda - \epsilon}{2+\epsilon+\epsilon\lambda}, \qquad |\lambda| \leq 1.$$

Then for every $\epsilon \geq 0$ one has $g_\epsilon \in \mathrm{A}^1_+(\mathbb{D})$ and $\|g_\epsilon\|_{\mathrm{A}^1_+(\mathbb{D})} = 1$.

Proof. Note that

$$\begin{aligned} g_\epsilon(\lambda) &= -1 + \frac{2(1+\lambda)}{2+\epsilon+\epsilon\lambda} \\ &= -1 + \frac{2}{2+\epsilon} + \frac{4\lambda}{(2+\epsilon)(2+\epsilon+\epsilon\lambda)} \\ &= -\frac{\epsilon}{2+\epsilon} + \frac{4\lambda}{(2+\epsilon)^2}\sum_{n=0}^\infty \frac{(-1)^n\epsilon^n\lambda^n}{(2+\epsilon)^n}, \qquad |\lambda| \leq 1. \end{aligned}$$

Hence

$$\begin{aligned} \|g_\epsilon\|_{\mathrm{A}^1_+(\mathbb{D})} &= \frac{\epsilon}{2+\epsilon} + \frac{4}{(2+\epsilon)^2}\sum_{n=0}^\infty \frac{\epsilon^n}{(2+\epsilon)^n} \\ &= \frac{\epsilon}{2+\epsilon} + \frac{2}{2+\epsilon} = 1. \end{aligned}$$

□

Now we are ready to prove our main result. Besides saying that a convex power series of a Ritt operator is Ritt, it shows that a convex power series preserves Stolz type. Moreover we have a control over the angle of the Ritt operator given by the series.

In the course of our proof we first show that $1 - h(T) = \mathbf{h}(\mathcal{C}(T))$ in a "simple" case when T is Ritt and $1 - T$ is invertible. Then the sectoriality estimate for $\mathbf{h}(\mathcal{C}(T))$ given by Theorem 6.7 transfers to that for $1 - h(T)$. The general case then follows by approximation, and the uniformity of the constant in (6.4) with respect to a family approximating T appears to be indispensable.

Theorem 6.7. *Let h be defined by (6.13), and let T be a Ritt operator on X . Then there exists $\omega \in [0, \pi/2)$ such that $\mathcal{C}(T) \in \mathrm{Sect}(\omega)$, and $h(T)$ is a Ritt operator on X with the same angle ω. Moreover, if T is of Stolz type σ, then $h(T)$ has Stolz type σ as well.*

Proof. Since by assumption T is Ritt, T is power bounded and $\sigma(T) \subset \mathbb{D} \cup \{1\}$.

Assume first that $1 \notin \sigma(T)$, so that $\sigma(T) \subset \mathbb{D}$. Let $\mathcal{C}$ be the Cayley transform defined by (4.12), and set for shorthand $A := \mathcal{C}(T)$. Clearly $A \in \mathcal{L}(X)$, and Proposition 4.7 implies that $A \in \mathrm{Sect}(\omega)$ for some $\omega \in [0, \pi/2)$. Moreover, $\ker(A) = \{0\}$. Let us first prove that

$$1 - h(T) = \mathbf{h}(A), \tag{6.20}$$

where $\mathbf{h}(A)$ is defined by the extended holomorphic calculus and $h(T)$ is given by the $\mathrm{A}^1_+(\mathbb{D})$-calculus.

By the definition of the extended holomorphic calculus,

$$\mathbf{h}(A) = A^{-1}(1 + A)^2(\mathbf{h} \cdot \tau)(A). \tag{6.21}$$

If $\omega' \in (\omega, \pi/2)$, then using (4.14) and Cauchy's theorem, we obtain

$$\begin{aligned}
(\mathbf{h} \cdot \tau)(A) &= \frac{1}{2\pi i} \int_{\partial \Sigma_{\omega'}} \frac{\lambda \, \mathbf{h}(\lambda)}{(\lambda + 1)^2} (\lambda - A)^{-1} \, d\lambda \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_{\omega'}} \frac{\lambda}{(\lambda + 1)^2} \sum_{n=0}^{\infty} c_n \left[1 - \left(\frac{1 - \lambda}{1 + \lambda} \right)^n \right] (\lambda - A)^{-1} \, d\lambda \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} c_n \int_{\partial \Sigma_{\omega'}} \frac{\lambda}{(\lambda + 1)^2} \left[1 - \left(\frac{1 - \lambda}{1 + \lambda} \right)^n \right] (\lambda - A)^{-1} \, d\lambda \\
&= \sum_{n=0}^{\infty} c_n A(A + 1)^{-2} \left[1 - ((1 - A)(1 + A)^{-1})^n \right] \\
&= A(A + 1)^{-2} \sum_{n=0}^{\infty} c_n (1 - T^n) \\
&= A(A + 1)^{-2}(1 - h(T)).
\end{aligned}$$

This and (6.21) imply (6.20).

From (6.20) and Theorem 6.4 it follows that $1 - h(T) \in \mathrm{Sect}(\omega)$, where $\omega \in [0, \pi/2)$ is a sectoriality angle of $A = \mathcal{C}(T)$. Hence, by the spectral mapping theorem (Theorem 5.4) and Theorem 4.1, $h(T)$ is a Ritt operator of angle ω. Thus the first statement of the theorem is proved if T is Ritt such that $1 - T$ is invertible.

Let now $1 \in \sigma(T)$. Then consider the approximation family $(T_\epsilon)_{\epsilon \in (0,1)} \subset \mathcal{L}(X)$ given by

$$T_\epsilon := g_\epsilon(T) = ((2 - \epsilon)T - \epsilon)(2 + \epsilon + \epsilon T)^{-1}, \qquad \epsilon \in (0, 1). \tag{6.22}$$

Observe that since T is Ritt, the spectral mapping theorem for the (standard) Riesz-Dunford functional calculus and simple geometric considerations

imply that

$$\sigma(T_\epsilon) \subset \mathbb{D}.$$

Thus $h(T_\epsilon)$ is well-defined in the $\mathrm{A}^1_+(\mathbb{D})$-calculus.

Furthermore, note that for every $\epsilon \in (0,1)$,

$$\mathcal{C}(T_\epsilon) = (1+\epsilon - T + \epsilon T)(1+T)^{-1} = \mathcal{C}(T) + \epsilon, \tag{6.23}$$

so that $\mathcal{C}(T_\epsilon) \in \mathrm{Sect}(\omega)$. Moreover,

$$\lim_{\epsilon\to 0} \|T - T_\epsilon\| = \|\epsilon(1+T)^2(2+\epsilon+\epsilon T)^{-1}\| = 0. \tag{6.24}$$

Hence for every $n \in \mathbb{Z}_+$,

$$\lim_{\epsilon\to 0} \|T^n - T_\epsilon^n\| = 0. \tag{6.25}$$

Since, in view of Lemma 6.6,

$$\|h(T_\epsilon)\| = \|h(g_\epsilon(T))\|_{\mathrm{A}^1_+(\mathbb{D})} \le \sum_{n=0}^{\infty} c_n \|g_\epsilon(T)\|^n_{\mathrm{A}^1_+(\mathbb{D})} = Mh(1) = M,$$

where $M := \sup_{n\ge 0} \|T^n\|$, the dominated convergence theorem gives

$$\lim_{\epsilon\to 0} \|h(T) - h(T_\epsilon)\| = 0. \tag{6.26}$$

(Note that we do need any composition rule here.)

According to the first part of the proof and (6.23), for each $\epsilon \in (0,1)$,

$$1 - h(T_\epsilon) = \mathbf{h}(\mathcal{C}(T_\epsilon)) = \mathbf{h}(\mathcal{C}(T)+\epsilon) = \mathbf{h}(A+\epsilon). \tag{6.27}$$

Next we use Theorem 6.4 again. The estimate given by (6.4) and (6.5) and Proposition 5.2 yield

$$-\sigma(\mathbf{h}(A+\epsilon)) \subset \mathbb{C} \setminus \Sigma_\omega \tag{6.28}$$

and

$$\|(z + \mathbf{h}(A+\epsilon))^{-1}\| \le \frac{M_{\omega'}}{|z|}, \qquad z \in \Sigma_{\omega'}, \tag{6.29}$$

for all $\epsilon \in (0,1)$ and $\omega' > \omega$, where $M_{\omega'}$ *does not depend* on ϵ. Now due to (6.28) and (6.27) we have $\sigma(h(T_\epsilon)) \subset 1 - \overline{\Sigma}_\omega$, so that by (6.26),

$$\sigma(h(T)) \subset 1 - \overline{\Sigma}_\omega.$$

Moreover, (6.27) and (6.29) imply

$$\|(z - 1 + h(T_\epsilon))^{-1}\| \le \frac{M_{\omega'}}{|z|}, \qquad z \in \Sigma_{\omega'}, \tag{6.30}$$

for all $\epsilon \in (0,1)$ and $\omega' > \omega$. Then, by (6.26) and (6.30),

$$\lim_{\epsilon\to 0} (z - 1 + h(T_\epsilon))^{-1} = (z - 1 + h(T))^{-1}$$

strongly in $\mathcal{L}(X)$ for each $z \in \Sigma_\omega$. Therefore, for all $\omega' > \omega$ and $z \in \Sigma_{\omega'}$,

$$\|(z - 1 + h(T))^{-1}\| \le \liminf_{\epsilon\to 0} \|(z - 1 + h(T_\epsilon))^{-1}\| \le \frac{M_{\omega'}}{|z|}.$$

In other words, $h(T)$ is a Ritt operator of angle ω.

Let us now prove the claim about Stolz type. Assume that T is a Ritt operator of Stolz type σ. Then by the preceding part of the proof and Proposition 4.5 the operator $h(T)$ is Ritt of angle $\alpha = \arccos(1/\sigma)$. Hence for any $\beta \in (\alpha, \pi)$,

$$\|(z - h(T))^{-1}\| \le \frac{C_\beta}{|z-1|}, \qquad 1 - z \notin \overline{\Sigma}_\beta. \tag{6.31}$$

On the other hand, by Theorem 5.4 and Proposition 4.4, we conclude that

$$\sigma(h(T)) \subset \overline{S}_\sigma. \tag{6.32}$$

Let $\delta > \sigma$ be fixed, and let $\sigma_0 := (\sigma + \delta)/2$ and $\alpha_0 := \arccos(1/\sigma_0)$. Then, by (6.31) and (4.5), there is $C_{\alpha_0} \ge 1$ such that

$$\|(z - h(T))^{-1}\| \le \frac{C_{\alpha_0}}{|z-1|}, \qquad 1 - z \notin \overline{\Sigma}_{\alpha_0}.$$

If $z \in \mathbb{D} \setminus S_\delta$ and $1 - z \in \overline{\Sigma}_{\alpha_0}$, then a simple calculation shows that

$$\mathrm{Re}(1-z) \ge \frac{2\delta(\delta - \sigma_0)}{(\delta^2 - 1)\sigma_0^2}.$$

Therefore, the distance between $(\mathbb{D} \setminus S_\delta) \cap (1 - \overline{\Sigma}_{\alpha_0})$ and S_σ is positive, and, in view of (6.32),

$$\|(z - h(T))^{-1}\| \le \tilde{C}, \qquad z \notin S_\delta, \quad 1 - z \in \overline{\Sigma}_{\alpha_0},$$

for some $\tilde{C} > 0$. Taking into account

$$\mathbb{C} \setminus S_\delta \subset \{z \in \mathbb{C} : \, 1 - z \notin \overline{\Sigma}_{\alpha_0}\} \cup \{z \in \mathbb{C} : \, z \notin S_\delta, \quad 1 - z \in \overline{\Sigma}_{\alpha_0}\},$$

we conclude that the operator $h(T)$ satisfies (4.9). As the choice of $\delta > \sigma$ is arbitrary, $h(T)$ is of Stolz type σ. $\square$

7. Hausdorff functions of Ritt operators: improving properties

As we mentioned in the introduction, our technique allows one to characterize improving properties of certain $\mathrm{A}^1_+(\mathbb{D})$-functions, namely of its subclass consisting of Hausdorff functions. In particular, the following simple geometric criterion holds.

Theorem 7.1. *Let h be a non-constant regular Hausdorff function, and let $\gamma \in (0, \pi/2)$ be fixed. The following statements are equivalent.*

(i) *One has*

$$1 - h(\lambda) \subset \overline{\Sigma}_\gamma, \qquad \lambda \in \mathbb{D}. \tag{7.1}$$

(ii) *For every Banach space X and every power bounded operator T on X the operator $h(T)$ is Ritt of angle γ.*

Proof. Let us first prove that (i) implies (ii). Let ψ be defined by

$$\psi(\lambda) := 1 - h(1-\lambda), \qquad \lambda \in \mathbb{D}, \tag{7.2}$$

and denote its extension to $\mathbb{C}\setminus(-\infty, 0]$ given by Proposition 3.8 by the same symbol. Thus, $\psi \in \mathcal{CBF}$ and by assumption

$$\psi(\lambda) \in \overline{\Sigma}_\gamma, \qquad \lambda \in \mathbb{D}_1 =: 1 + \mathbb{D} = \{\lambda \in \mathbb{C} : \, |\lambda - 1| < 1\}. \tag{7.3}$$

As for Theorem 6.7, the proof will be done in two steps. Suppose first that

$$\{-1, 1\} \not\subset \sigma(T). \tag{7.4}$$

If $\varphi(\lambda) := \frac{2\lambda}{\lambda+1}$, then $\varphi \in \mathcal{CBF}$, and $\psi \circ \varphi \in \mathcal{CBF}$ as a composition of complete Bernstein functions. Moreover, since $\varphi : \mathbb{C}_+ \to \mathbb{D}_1$, it follows from (7.3) that

$$(\psi \circ \varphi)(\mathbb{C}_+) \subset \overline{\Sigma}_\gamma.$$

Noting that

$$\varphi^{-1}(\lambda) = \frac{\lambda}{2-\lambda}, \quad \lambda \neq 2,$$

and setting $Q := 1 - T$, we conclude by Proposition 4.6 that

$$\varphi^{-1}(Q) = Q(2-Q)^{-1} = \mathcal{C}(T) \in \mathrm{Sect}(\pi/2),$$

and, by Theorem 1.1, b) and the composition rule (5.3), we obtain that

$$\psi(Q) = (\psi \circ \varphi)(\varphi^{-1}(Q)) = (\psi \circ \varphi)(\mathcal{C}(T)) \in \mathrm{Sect}(\gamma).$$

Furthermore, by Lemma 5.6,

$$\psi(Q) = 1 - h(T), \tag{7.5}$$

where $\psi(Q)$ is defined in the extended holomorphic functional calculus, and $h(T)$ is given by $\mathrm{A}^1_+(\mathbb{D})$-calculus.

Observe that by (3.21), $h(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$. Moreover, if $\lambda \in \overline{\mathbb{D}}$ and $|h(\lambda)| = 1$, then

$$1 = \left|\sum_{k=0}^{\infty} c_k \lambda^k\right| \le \sum_{k=0}^{\infty} c_k = 1,$$

and $\lambda = h(\lambda) = 1$, so that $h(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}$. From Theorem 5.4 it then follows that $\sigma(h(T)) \subset \mathbb{D} \cup \{1\}$, and by Theorem 6.5 we conclude that $h(T)$ is a Ritt operator of angle γ. Thus the statement is proved for power bounded T such that (7.4) holds.

If (7.4) does not hold, then we let $\sup_{n\ge 0} \|T^n\| := M$ and consider the family of bounded linear operators $T_\epsilon := (1-\epsilon)T, \epsilon \in (0,1)$. Clearly, $\sigma(T_\epsilon) \subset (1-\epsilon)\overline{\mathbb{D}}$ for each $\epsilon \in (0,1)$ and

$$\|(z - T_\epsilon)^{-1}\| = (1-\epsilon)^{-1}\|((1-\epsilon)^{-1}z - T)^{-1}\| \le \frac{M}{|z| - 1}, \qquad |z| > 1. \tag{7.6}$$

Hence, by the first step, if $Q_\epsilon = 1 - T_\epsilon$ then $\psi(Q_\epsilon) \in \mathrm{Sect}(\gamma)$. Moreover, as $\|T_\epsilon\| \le \|T\|$ and (7.6) holds, by (4.15) and Theorem 6.5 again we infer that for each $\beta \in (\gamma, \pi)$ there exists M_β *independent* of $\epsilon > 0$ such that

$$\|(z + \psi(Q_\epsilon))^{-1}\| \le \frac{M_\beta}{|z|}, \qquad z \in \Sigma_{\pi-\beta}. \tag{7.7}$$

Thus, taking into account (7.5), we infer that

$$\|(1-h(T_\epsilon)+z)^{-1}\| \le \frac{M_\beta}{|z|}, \qquad z \in \Sigma_{\pi-\beta}, \tag{7.8}$$

for each $\beta \in (\gamma,\pi)$. Moreover, since $h \in \mathrm{A}^1_+(\mathbb{D})$, we have

$$\lim_{\epsilon\to 0} \|h(T_\epsilon) - h(T)\| = 0, \tag{7.9}$$

hence Proposition 5.4 and (5.7) yield

$$\sigma(h(T)) \subset 1 - \overline{\Sigma}_\gamma. \tag{7.10}$$

Now, combining (7.8) and (7.9), we infer that

$$(1-h(T)+z)^{-1} = \lim_{\epsilon\to 0}(1-h(T_\epsilon)+z)^{-1} \tag{7.11}$$

strongly in $\mathcal{L}(X)$ for every nonzero $z \in \mathbb{C}\setminus\overline{\Sigma}_{\pi-\gamma}$. Therefore,

$$\|(1-h(T)+z)^{-1}\| \le \liminf_{\epsilon\to 0}\|(1-h(T_\epsilon)+z)^{-1}\| \le \frac{M_\beta}{|z|}, \qquad z \in \Sigma_{\pi-\beta}, \tag{7.12}$$

for every $\beta \in (\gamma,\pi)$. Now, (7.10) and (7.12) imply the claim.

The implication (ii) $\Rightarrow$ (i) is proved in [23, p. 1728]. It suffices to consider the multiplication operator $(Tf)(\lambda) = \lambda f(\lambda), \lambda \in \mathbb{D}$, on $X = C(\overline{\mathbb{D}})$ and to use the fact that if $h(T)$ is Ritt of angle ω, then the multiplication semigroup $(e^{(1-h(T))t})_{t\ge 0}$:

$$(e^{(1-h(T))t}f)(\lambda) = e^{(1-h(\lambda))t}f(\lambda), \qquad \lambda \in \overline{\mathbb{D}},$$

is sectorially bounded on $\Sigma_{\omega'}$ for every $\omega' > \omega$. □

As an illustration of Theorem 7.1, we show how several main results from [23] can be obtained by our technique and answer a question posed in [23]. Moreover, we show that Theorem 7.1 provides "geometrical" improvements of the results from [23].

Example 7.2. By Example 3.9, a) the function $h_\alpha(\lambda) = 1-(1-\lambda)^\alpha$ is regular Hausdorff for every $\alpha \in (0,1)$. Moreover,

$$(1-h_\alpha)(\mathbb{D}) \subset \overline{\Sigma}_{\alpha\pi/2}.$$

Thus, for any power-bounded operator T on X and any $\alpha \in (0,1)$, the operator $h_\alpha(T)$ is Ritt of angle $\alpha\pi/2$. Clearly, Theorem 7.1 extends [23, Theorem 1.1 and Theorem 4.3], where the special case of Theorem 7.1 for $h_\alpha, \alpha \in (0,1)$, was considered.

Example 7.3. For $\alpha > 0$ define

$$L_{1+\alpha}(\lambda) := \frac{1}{\zeta(1+\alpha)}\sum_{n=1}^\infty \frac{\lambda^n}{n^{1+\alpha}},$$

where ζ is the zeta function. The functions $L_{1+\alpha}$ arise as generating functions for so-called zeta-probabilities [23]. They are also related to fractional polylogarithms, see e.g. [17]. As discussed in [23], they appear to be useful in probability theory.

Clearly, $L_{1+\alpha} \in \mathrm{A}^1_+(\mathbb{D})$ for every $\alpha > 0$. It was proved in [23, Theorem 1.2 and Theorem 4.4] that for every $\alpha \in (0,1)$ and any power-bounded operator T on X the operator $L_{1+\alpha}(T)$ is Ritt. We show that this result follows from Theorem 7.1 and, moreover, we are able to provide a bound for the corresponding angle of $L_\alpha(T)$. To this aim, note that since for every $k \in \mathbb{N}$,

$$\frac{1}{k^{1+\alpha}} = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty e^{-kt} t^\alpha \, dt = \frac{1}{\Gamma(1+\alpha)} \int_0^1 (\log(1/s))^\alpha s^{k-1} \, ds,$$

the function $L_{1+\alpha}$ is Hausdorff for each $\alpha \in (0,1)$. Moreover, by [26, p 29, 1.11 (8)] we have

$$\begin{aligned} \zeta(1+\alpha) L_{1+\alpha}(\lambda) =& \lambda \Phi(\lambda, 1+\alpha; 1) \\ =& \zeta(1+\alpha) + \Gamma(-\alpha)(\log(1/\lambda))^\alpha + \mathrm{O}(|\lambda - 1|), \end{aligned} \tag{7.13}$$

as $\lambda \to 1, \lambda \in \overline{\mathbb{D}}$, where Φ is the Lerch zeta function (see e.g. [1]). Taking into account the inequality

$$|L_{1+\alpha}(\lambda)| \le 1, \qquad \lambda \in \overline{\mathbb{D}},$$

and (7.13), we infer that

$$1 - L_{1+\alpha}(\lambda) \in \overline{\Sigma}_\beta, \qquad \lambda \in \overline{\mathbb{D}},$$

for some $\beta = \beta(\alpha) \in (0, \pi/2)$. Hence, by Theorem 7.1, for any power-bounded operator T on X and any $\alpha \in (0,1)$ the operator $L_{1+\alpha}(T)$ is Ritt of angle $\beta(\alpha)$.

However, we can be more precise here. By combining Theorem 7.1 with Proposition 10.1 from Appendix B, we conclude that $L_{1+\alpha}(T)$ is of angle $\beta(\alpha) = \alpha\pi/2$.

Example 7.4. Let for a fixed $\epsilon \in (0,1)$,

$$h_\epsilon(\lambda) := 1 - \frac{1}{\epsilon} \int_0^\epsilon (1-\lambda)^\alpha \, d\alpha = 1 - \frac{(1-\lambda)^\epsilon - 1}{\epsilon \log(1-\lambda)}, \qquad \lambda \in \mathbb{D}. \tag{7.14}$$

The function h_ϵ extends holomorphically to $\mathbb{C} \setminus (-\infty, 0]$, and denoting the extension by the same symbol we infer that

$$f_\epsilon(\lambda) := 1 - h_\epsilon(1-\lambda) = \frac{\lambda^\epsilon - 1}{\log \lambda^\epsilon}, \qquad \lambda > 0,$$

belongs to $\mathcal{CBF}$ as a composition of the complete Bernstein functions $(\lambda - 1)/\log\lambda$ and λ^ϵ. If $\lambda \in \mathbb{C}_+$, then $\lambda^\epsilon \in \overline{\Sigma}_{\epsilon\pi/2}$, hence in view of the integral representation in (7.14), $f_\epsilon(\mathbb{C}_+) \in \overline{\Sigma}_{\epsilon\pi/2}$. Thus, by Example 3.9, b), the function h_ϵ is regular Hausdorff and

$$(1 - h_\epsilon)(\mathbb{D}) \subset \overline{\Sigma}_{\epsilon\pi/2}.$$

By Theorem 7.1, we conclude that if T is a power-bounded operator on X, then $h_\epsilon(T)$ is Ritt of angle $\epsilon\pi/2$. This settles a conjecture posed in [23, p. 1735]. Note that Theorem 5.1 in [23] can also be treated in a similar way, but we leave the details to the interested reader.

Note that the angle $\epsilon\pi/2$ given by Theorem 7.1 is not optimal. For instance, for a Hausdorff function h_1,

$$h_1(\lambda) = 1 - f_1(1-\lambda) = 1 + \frac{\lambda}{\log(1-\lambda)}, \qquad \lambda \in \mathbb{D},$$

we have by Lemma 10.2 from Appendix B:

$$(1-h_1)(\mathbb{D}) \subset \overline{\Sigma}_{\pi/3}.$$

So, by Theorem 7.1, we obtain that for any power-bounded operator T on X the operator $h_1(T)$ is Ritt of angle $\pi/3$.

Remark, finally, that [23] deals with the elements of $\ell_1(\mathbb{Z}_+)$ given by $\frac{1}{\epsilon}\int_0^\epsilon A_\alpha \, d\alpha$, where $A_\alpha \in \ell_1(\mathbb{Z}_+)$ is the sequence of Taylor coefficients of $(1-z)^\alpha$, rather than the generating function of $\frac{1}{\epsilon}\int_0^\epsilon A_\alpha \, d\alpha$. However, since $\mathrm{A}^1_+(\mathbb{D})$ and $\ell_1(\mathbb{Z}_+)$ are isometrically isomorphic as Banach algebras, both settings are, in fact, equivalent.

8. A remark on angles of Ritt operators

In this section, by means of a simple example, we illustrate the statement of Theorem 6.7 on angles of Ritt operators. To this aim, we introduce the following notation. Let the *minimal* angle $\alpha(T)$ of a Ritt operator T on X be defined as

$$\alpha(T) = \inf\{\alpha : T \text{ is Ritt of angle } \alpha\}.$$

We show that there exists a Ritt operator T with the minimal angle $\alpha(T)$ and a function h satisfying (6.13) such that $h(T)$ is a Ritt operator with the minimal angle $\alpha(h(T))$ greater than $\alpha(T)$. Moreover, the difference $\alpha(h(T)) - \alpha(T)$ can be arbitrarily close to $\pi/2$. Thus discrete subordination does not, in general, preserve angles of Ritt operators. This justifies, in particular, the use of the Cayley transform and Stolz types in the study of permanence properties for discrete subordination.

In the notation of Theorem 6.7, let

$$h(\lambda) := \lambda^2, \qquad \lambda \in \mathbb{D},$$

and for $\varphi \in (0, \pi/2)$ and $\rho \in (0, 2\cos\varphi)$ set $\lambda = 1 - \rho e^{i\varphi}$. Note that

$$\begin{aligned} \frac{|\mathrm{Im}\,(1-h(\lambda))|}{\tan\varphi \,\mathrm{Re}\,(1-h(\lambda))} &= \frac{|2\sin\varphi - \rho\sin(2\varphi)|}{\tan\varphi\,(2\cos\varphi - \rho\cos(2\varphi))} \\ &= \frac{|1-2t\cos^2\varphi|}{1-t\cos(2\varphi)}, \end{aligned} \tag{8.1}$$

where $t = \rho/(2\cos\varphi)$ so that $t \in (0,1)$.

On the other hand,

$$\frac{|\mathrm{Im}\,\mathcal{C}(\lambda)|}{\tan\varphi\,\mathrm{Re}\,\mathcal{C}(\lambda)} = \frac{2\cos\varphi}{2\cos\varphi - \rho} = \frac{1}{1-t}. \tag{8.2}$$

Let $X = L^2((0,1))$. For fixed $\varphi \in (0,\pi/2)$ and $\delta \in (0,1)$ define the operator $T_{\varphi,\delta}$ on X by

$$(T_{\varphi,\delta} y)(t) := (1 - 2t\delta\cos\varphi e^{i\varphi}) y(t), \qquad y \in L^2((0,1)). \tag{8.3}$$

Theorem 8.1. *Let X and $T_{\varphi,\delta} \in \mathcal{L}(X)$ be defined by (8.3). Then*

(i) *$T_{\varphi,\delta}$ is a Ritt operator of the minimal angle $\alpha(T_{\varphi,\delta}) = \varphi$.*
(ii) *for each $\epsilon \in (0,1)$ there exist $\varphi \in (0,\pi/2)$ and $\delta \in (0,1)$ such that $T^2_{\varphi,\delta}$ is a Ritt operator of the minimal angle $\alpha(T^2_{\varphi,\delta}) := \beta_{\varphi,\delta}$ satisfying*

$$\tan \beta_{\varphi,\delta} \geq \frac{\tan\varphi}{\epsilon}. \tag{8.4}$$

Hence, $\beta_{\varphi,\delta} - \tan\varphi$ can be arbitrarily close to $\pi/2$.
(iii) *If $\gamma_{\varphi,\delta}$ is the minimal sectoriality angle of $\mathcal{C}(T_{\varphi,\delta})$, then $\gamma_{\varphi,\delta}$ can be arbitrarily close to $\beta_{\varphi,\delta}$.*

Remark 8.2. Recall that $T^2_{\varphi,\delta}$ is Ritt of angle $\gamma_{\varphi,\delta}$ by Theorem 6.7.

Proof. A direct calculation shows that $T_{\varphi,\delta}$ is a Ritt operator of the minimal angle φ, and thus proves (i). Hence, by (8.2),

$$\tan \gamma_{\varphi,\delta} = \frac{\tan\varphi}{1-\delta}. \tag{8.5}$$

So, $\gamma_{\varphi,\delta} > \varphi$.

Let

$$v_\varphi(t) := \frac{|1 - 2t\cos^2\varphi|}{1 - t\cos(2\varphi)},$$

where $t = \rho/(2\cos\varphi)$. Observe that by (8.1),

$$\tan\beta_{\varphi,\delta} = \tan\varphi \sup_{t\in(0,1)} v_\varphi(\delta t) \geq v_\varphi(\delta)\tan\varphi.$$

Hence from (8.5) it follows that

$$\frac{\tan\beta_{\varphi,\delta}}{\tan\gamma_{\varphi,\delta}} \geq v_\varphi(\delta) = \frac{(1-\delta)(\delta\cos(2\varphi)+\delta-1)}{1-\delta\cos(2\varphi)}.$$

Let $\epsilon \in (0,1)$ be fixed and let $\varphi = \arccos \delta^{\epsilon/2}/2$. Then

$$\begin{aligned}\limsup_{\delta\to 1} \frac{\tan\beta_{\varphi,\delta}}{\tan\gamma_{\varphi,\delta}} &\geq \limsup_{\delta\to 1} \frac{(1-\delta)(\delta^{1+\epsilon/2}+\delta-1)}{1-\delta^{1+\epsilon/2}} \\ &= 1/(1+\epsilon/2) \\ &> 1-\epsilon.\end{aligned}$$

This shows (iii). Now if $\delta \in (0,1)$ is such that

$$\frac{\tan\beta_{\varphi,\delta}}{\tan\gamma_{\varphi,\delta}} > 1-\epsilon \quad \text{and} \quad 1-\delta < (1-\epsilon)\epsilon,$$

then

$$\tan\beta_{\varphi,\delta} > (1-\epsilon)\tan\gamma_{\varphi,\delta} = \frac{1-\epsilon}{1-\delta}\tan\varphi \geq \frac{\tan\varphi}{\epsilon},$$

and (ii) follows. $\square$

9. Appendix A

In this appendix we collect several technical estimates used in previous sections. The lemma below is crucial in the characterization of Ritt operators in terms of Stolz domains given in Proposition 4.2.

Lemma 9.1. *Let $z \in \mathbb{D}$ and let*

$$Q_z(\varphi) := \frac{|z - e^{i\varphi}|}{|1 - e^{i\varphi}|}, \qquad \varphi \in [0, 2\pi).$$

Then

$$m_z := \min_{\varphi \in [0,2\pi)} Q_z(\varphi) = \frac{1 - |z|^2}{2|1 - z|}. \tag{9.1}$$

Proof. Let $z = re^{i\alpha}$, where $r \in [0, 1)$ and $\alpha \in [0, 2\pi)$. Then, setting $\psi := \varphi/2 \in (0, \pi)$, we obtain

$$\begin{aligned}
4Q_z^2(\varphi) =& \frac{r^2 + 1 - 2r\cos\alpha + 4r\cos\alpha\sin^2\psi - 4r\sin\alpha\sin\psi\cos\psi}{\sin^2\psi} \\
=& |1 - z|^2(1 + \cot^2\psi) + 4r\cos\alpha - 4r\sin\alpha\cot\psi \\
=& \left(|1 - z|\cot\psi - \frac{2r\sin\alpha}{|1 - z|}\right)^2 - \frac{4r^2\sin^2\alpha}{|1 - z|^2} + |1 - z|^2 + 4r\cos\alpha.
\end{aligned}$$

Hence a simple calculation shows that

$$\begin{aligned}
4m_z^2 =& -\frac{4r^2\sin^2\alpha}{|1 - z|^2} + |1 - z|^2 + 4r\cos\alpha \\
=& \frac{(1 - |z|^2)^2}{|1 - z|^2},
\end{aligned}$$

and (9.1) follows. □

The next simple lemma is instrumental in the proof of Theorem 6.3.

Lemma 9.2. *For all $\gamma \in [0, \pi)$, $\beta \in [0, \pi)$ such that $\gamma + \beta < \pi$,*

$$|z + \lambda| \ge \cos((\gamma + \beta)/2)\,(|z| + |\lambda|), \quad z \in \overline{\Sigma}_\gamma, \quad \lambda \in \overline{\Sigma}_\beta. \tag{9.2}$$

Proof. Note first that if $\beta \in (-\pi, \pi)$ and $s > 0$ then

$$|1 + se^{i\beta}|^2 = 1 + s^2 + 2s\cos\beta \ge \cos^2(\beta/2)(1 + s)^2. \tag{9.3}$$

Let now $\gamma > 0, \quad \beta > 0, \quad \gamma + \beta < \pi$, and

$$z = re^{i\gamma_0} \in \overline{\Sigma}_\gamma, \quad \lambda = \rho e^{i\beta_0} \in \overline{\Sigma}_\beta, \quad |\gamma_0| \le \gamma, \quad |\beta_0| \le \beta.$$

Then, using (9.3), we obtain

$$|z + \lambda| = r|1 + r^{-1}\rho e^{i(\beta_0 - \gamma_0)}| \ge \cos((\beta_0 - \gamma_0)/2)\,(|z| + |\lambda|).$$

From this, since

$$|\beta_0 - \gamma_0| \le \beta + \gamma \in [0, \pi), \quad \text{and} \quad \cos((\beta_0 - \gamma_0)/2) \ge \cos((\beta + \gamma)/2),$$

it follows that

$$|z + \lambda| \ge \cos((\beta + \gamma)/2)\,(|z| + |\lambda|), \quad z \in \overline{\Sigma}_\gamma, \quad \lambda \in \overline{\Sigma}_\beta. \tag{9.4}$$

□

Now we turn to the proof of Lemma 3.6 which was essential in our arguments leading to Theorems 6.4 and 6.7. The proof of this lemma is based on several auxiliary estimates.

Lemma 9.3. *For all $R > 0$ and $\beta \in (-\pi/2, \pi/2)$ there exists*
$$b = b(\beta, R) \in (0, \min\{1, 1/(2R)\}),$$
such that
$$\left|\frac{1 - re^{i\beta}}{1 + re^{i\beta}}\right| \le \frac{1 - br}{1 + br}, \qquad r \in (0, R), \tag{9.5}$$
and
$$\frac{1}{|1 + re^{i\beta}|^2} \le \frac{1}{(1 + br)^2}, \qquad r \in (0, R). \tag{9.6}$$
Moreover, for each fixed $R > 0$, $b = b(\cdot, R)$ can be arranged to be decreasing function on $(0, \pi/2)$.

Proof. The estimate (9.5) is equivalent to
$$(1 + r^2 - 2r\cos\beta)(1 + br)^2 \le (1 + r^2 + 2r\cos\beta)(1 - br)^2.$$
Rearranging terms yields
$$(1 + r^2)[(1 + br)^2 - (1 - br)^2] \le 2r\cos\beta[(1 + br)^2 + (1 - br)^2],$$
or
$$\alpha(1 + r^2) \le \cos\beta(1 + b^2r^2), \qquad r \in (0, R).$$
The last inequality holds if
$$b = b(\beta, R) = \cos\beta/(1 + R^2) \in (0, \min\{1, 1/(2R)\}). \tag{9.7}$$
Moreover, if b is defined by (9.7), then
$$(1 + br)^2 \le (1 + r\cos\beta)^2 \le 1 + r^2 + 2r\cos\beta = |1 + re^{i\beta}|^2, \qquad r > 0,$$
and (9.6) follows.

Given the definition (9.7), the last claim is straightforward. □

Define for each $n \in \mathbb{N}$
$$h_n(\lambda) := \left(\frac{1 - \lambda}{1 + \lambda}\right)^n, \qquad q_n(\lambda) := 1 - h_n(\lambda), \qquad \lambda \ne -1. \tag{9.8}$$

Note that $|h_n(\lambda)| \le 1$, $|q_n(\lambda)| \le 2$, $\lambda \in \overline{\mathbb{C}}_+$, and that q_n maps $\overline{\mathbb{C}}_+$ into $\overline{\mathbb{C}}_+$. Moreover, for every $n \in \mathbb{N}$,
$$\begin{aligned} h_n'(\lambda) &= -2n\frac{h_{n-1}(\lambda)}{(1 + \lambda)^2}, \\ h_n''(\lambda) &= \frac{4n}{(1 + \lambda)^4}[(n - 1)h_{n-2}(\lambda) + (1 + \lambda)h_{n-1}(\lambda)], \end{aligned}$$
where we set $h_0(\lambda) \equiv 1$ and $h_{-1}(\lambda) \equiv 0$. In particular, the functions h_n and $-h_n'$ are positive and decreasing on $(0, 1)$ for each $n \in \mathbb{N}$.

Lemma 9.4. *For all $\beta \in (0, \pi/2)$ and $R > 0$ there exists $b = b(\beta, R) \in (0, \min\{1, 1/(2R)\})$ such that for every $n \in \mathbb{N}$,*

$$|\mathrm{Im}\, h_n(re^{i\beta})| \le -\frac{\pi}{2} r h_n'(br), \qquad r \in (0, R).$$

Proof. Let $\beta \in (0, \pi/2)$ and $R > 0$ be fixed. For every $r \in (0, R)$,

$$\begin{aligned}\mathrm{Im}\, h_n(re^{i\beta}) = \frac{h_n(re^{i\beta}) - h_n(re^{-i\beta})}{2i} &= \frac{1}{2i}\int_{-\beta}^{\beta} \frac{dh_n(re^{i\gamma})}{d\gamma}\, d\gamma \\ &= -nr \int_{-\beta}^{\beta} \frac{h_{n-1}(re^{i\gamma})}{(1+re^{i\gamma})^2}\, e^{i\gamma}\, d\gamma.\end{aligned}$$

Let $b_\gamma = b(\gamma, R)$ and $\gamma \in (0, \beta]$ be given by Lemma 9.3 (see (9.7)). Then, using Lemma 9.3 and the monotonicity of $-h_n'$ on $(0,1)$, we obtain for each $r \in (0, R)$:

$$\begin{aligned}|\mathrm{Im}\, h_n(re^{i\beta})| &\le 2\beta nr \sup_{\gamma\in(0,\beta)} \frac{|h_{n-1}(re^{i\gamma})|}{|1+re^{i\gamma}|^2} \\ &\le 2\beta nr \sup_{\gamma\in(0,\beta)} \frac{h_{n-1}(b_\gamma r)}{(1+b_\gamma r)^2} \\ &\le 2\beta nr\, \frac{h_{n-1}(b_\beta r)}{(1+b_\beta r)^2} \\ &= -\beta r h_n'(b_\beta r) \\ &\le -\frac{\pi}{2} r h_n'(b_\beta r).\end{aligned}$$

□

Now Lemma 3.6 follows directly from Lemma 9.4. Indeed, if

$$c_n \ge 0,\ n \ge 0, \qquad \sum_{n=0}^{\infty} c_n = 1,$$

and

$$\mathbf{h}(\lambda) := 1 - \sum_{n=0}^{\infty} c_n h_n(\lambda) = \sum_{n=0}^{\infty} c_n q_n(\lambda), \qquad \lambda \in \mathbb{C}_+,$$

then, by Lemma 9.4, for all $\beta \in (0, \pi/2)$ and $R > 0$, there exists

$$b = \cos\beta/(1+R^2) \in (0, \min\{1, 1/(2R)\})$$

such that

$$|\mathrm{Im}\, \mathbf{h}(re^{i\beta})| \le \frac{\pi}{2} r \mathbf{h}'(br), \qquad r \in (0, R).$$

In other words, $\mathbf{h} \in \mathcal{D}_{\pi/2}(0,1)$ with $m = \pi/2$ and b as above.

Finally, if $\lambda \in \mathbb{C}_+$, then

$$\mathrm{Re}\, \mathbf{h}(\lambda) \ge 1 - \sum_{k=0}^{\infty} c_k \left|\frac{1-\lambda}{1+\lambda}\right|^k \ge 0,$$

and, since $\mathbf{h}((0,\infty)) \subset [0, \infty)$, we conclude that $\mathbf{h} \in \mathcal{NP}_+$.

10. Appendix B

In this appendix, we prove several estimates which allowed us to obtain additional, geometric information on Hausdorff functions of Ritt operators in Section 7. We start with the proposition needed in Example 7.3.

Proposition 10.1. *For every* $\alpha \in (0,1)$,

$$L_{1+\alpha}(1) - L_{1+\alpha}(\lambda) \in \overline{\Sigma}_{\alpha\pi/2}, \qquad \operatorname{Re}\lambda \le 1. \tag{10.1}$$

Proof. Recall that by [26, p.27, 1.11(3)], for every $\alpha \in (0,1)$,

$$L_{1+\alpha}(\lambda) = \frac{\lambda}{\Gamma(1+\alpha)} \int_0^1 \frac{\log^\alpha(1/s)\,ds}{1-s\lambda}, \quad \lambda \in \mathbb{C}_+ \setminus (1,\infty).$$

and by Proposition 3.8 we have

$$\psi(\lambda) := 1 - L_{1+\alpha}(1-\lambda) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{\lambda \log^\alpha(1+t)\,dt}{(\lambda+t)t}, \quad \lambda \in \mathbb{C}\setminus(-\infty,0]. \tag{10.2}$$

If now $\lambda = |\lambda|e^{i\delta}$, $\delta \in (-\pi,\pi)$, then setting $t = \lambda\tau$ and using Cauchy's theorem, we infer from (10.2) that

$$\psi(\lambda) = \int_0^{e^{-i\delta}\infty} \frac{\log^\alpha(1+\lambda\tau)\,d\tau}{(1+\tau)\tau} = \int_0^\infty \frac{\log^\alpha(1+\lambda\tau)\,d\tau}{(1+\tau)\tau}. \tag{10.3}$$

If, moreover, $\lambda \in \mathbb{C}_+$, then for any $\tau > 0$ we have

$$\log^\alpha(1+\lambda\tau) \in \Sigma_{\alpha\pi/2}, \quad \alpha \in (0,1).$$

Thus taking into account (10.3) we get $\psi(\mathbb{C}_+) \subset \overline{\Sigma}_{\alpha\pi/2}$, and thus (10.1). $\square$

Now we prove an auxiliary result which is crucial in Example 7.4.

Lemma 10.2. *If*

$$h(\lambda) = \frac{\lambda - 1}{\log\lambda}, \qquad \lambda \in \mathbb{C}_+,$$

then

$$h(\mathbb{D}_1) \subset \overline{\Sigma}_{\pi/3}, \qquad \mathbb{D}_1 = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}.$$

Proof. Recall that $h \in \mathcal{CBF}$, hence

$$\operatorname{Im} h(\lambda) > 0, \qquad h(\lambda) = \overline{h(\overline{\lambda})}, \quad \operatorname{Im}\lambda > 0, \tag{10.4}$$

Let us first prove the following claim:

$$h(\lambda) \in \overline{\Sigma}_{\pi/4}, \qquad \lambda \in \overline{\mathbb{D}}_+, \quad \mathbb{D}_+ := \mathbb{C}_+ \cap \mathbb{D}. \tag{10.5}$$

For every $s > 0$ we have

$$h(is) = \frac{is-1}{\log s + i\pi/2} = \frac{(\pi s/2 - \log s) + i(s\log s + \pi/2)}{(\log s)^2 + \pi^2/4}.$$

If $s \in (0,1)$, then

$$\frac{\operatorname{Im} h(is)}{\operatorname{Re} h(is)} = \frac{s\log s + \pi/2}{\pi s/2 - \log s} \le 1 \quad \text{and} \quad h(is) \in \overline{\Sigma}_{\pi/4}.$$

Moreover, for $s \in (0,1)$,

$$\frac{d}{ds}|h(is)|^2 = 2\frac{s\pi^2/4 - s^{-1}\log s}{((\log s)^2+\pi^2/4)^2} > 0.$$

So, $|h(i\cdot)|$ is a strictly increasing function on $(0,1)$, $|h(0)| = 0$, and $|h(i)| = \frac{2\sqrt{2}}{\pi}$. Next, for every $\beta \in (0,\pi/2)$,

$$\frac{\operatorname{Im} h(e^{i\beta})}{\operatorname{Re} h(e^{i\beta})} = \frac{1-\cos\beta}{\sin\beta} = \tan(\beta/2) \le 1 \quad \text{and} \quad h(e^{i\beta}) \in \overline{\Sigma}_{\pi/4}.$$

Moreover, if $\beta \in (0,\pi/2)$, then

$$\frac{d}{d\beta}|h(e^{i\beta})| = \frac{\beta\cos(\beta/2) - 2\sin(\beta/2)}{\beta^2} < 0.$$

Hence, $|h(e^{i\cdot})|$ is a strictly decreasing function on $(0,\pi/2)$, $|h(0)| = 1$, and $|h(i)| = \frac{2\sqrt{2}}{\pi} < 1$.

Now from (10.4) it follows that h maps $\partial\mathbb{D}_+$ into $\partial\overline{\Sigma}_{\pi/4}$ injectively, and the claim is proved.

Finally, since $h \in \mathcal{CBF}$, we have

$$h(\overline{\Sigma}_{\pi/3} \setminus \{0\}) \subset \overline{\Sigma}_{\pi/3}, \tag{10.6}$$

by (3.4). Taking into account $\mathbb{D}_1 \subset \overline{\Sigma}_{\pi/3} \cup \mathbb{D}_+$, and, using (10.5) and (10.6), we obtain

$$h(\mathbb{D}_1) \subset \overline{\Sigma}_{\pi/3} \cup \overline{\Sigma}_{\pi/4} = \overline{\Sigma}_{\pi/3}.$$

□

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Faculty of Mathematics and Computer Science, Nicolas Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland, and Institute of Telecommunications and Global, Information Space, National Academy of Sciences of Ukraine, Kiev, Ukraine

E-mail address: gomilko@mat.umk.pl

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-956 Warsaw, Poland

E-mail address: ytomilov@impan.pl